TUGAS AKHIR - SF 141501

# KAJIAN RUANG WAKTU KERR-NEWMAN DALAM GRAVITASI EINSTEIN

ANDIKA IRAWAN
NRP 1112100070

Dosen Pembimbing
Dr.rer.nat Bintoro Anang Subagyo

JURUSAN FISIKA
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UNDERGRADUATE THESIS - SF 141501

# STUDY OF KERR-NEWMAN SPACETIME IN EINSTEIN GRAVITY

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Surabaya, 2016

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**KAJIAN RUANG WAKTU KERR-NEWMAN DALAM GRAVITASI EINSTEIN**

# TUGAS AKHIR

Diajukan Guna Memenuhi Salah Satu Syarat
Memperoleh Gelar Sarjana Sains
pada
Bidang Studi Fisika Teori dan Filsafat Alam
Program Studi S1 Jurusan Fisika
Fakultas Matematika dan Ilmu Pengetahuan Alam
Institut Teknologi Sepuluh Nopember

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KEMENTERIAN RISET, TEKNOLOGI, DAN PENDIDIKAN TINGGI
INSTITUT TEKNOLOGI SEPULUH NOPEMBER
FAKULTAS MATEMATIKA DAN ILMU PENGETAHUAN ALAM
JURUSAN FISIKA

**SURABAYA**
**Januari 2016**

# KAJIAN RUANG WAKTU KERR-NEWMAN DALAM GRAVITASI EINSTEIN

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## ABSTRAK

Dalam laporan tugas akhir ini telah dikaji solusi dari persamaan Einstein-Maxwell. Metrik dari solusi ini disebut sebagai metrik Kerr-Newman yang berotasi dan bermuatan tidak sama dengan nol. Pada awalnya dibahas solusi statis dan simetri bola, yaitu Schwarzschild dan Reissner-Nordstrom yang masing-masing merupakan solusi yang tidak bermuatan dan bermuatan. Kemudian untuk mendapatkan solusi berotasi, ruang-waktu diputar menggunakan algoritma Newman-Janis sehingga didapatkan solusi Kerr dan Kerr-Newman. Di sini juga dikaji sifat-sifat disekitar *event horizon* metrik Kerr-Newman.

**Kata-Kunci**: Persamaan Medan Einstein, Singularitas, Kerr-Newman

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## STUDY OF KERR-NEWMAN SPACETIME IN EINSTEIN GRAVITY

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## ABSTRACT

In this report we studied the solution of Einstein-Maxwell equations. The metric of solutions known as Kerr-Newman metric which are rotating and having non zero charge. In the beginning we discussed static and spherically symmetric solution, $i.e.$ Schwarzhild and Reisnerr-Nordstrom that are charged and non-charged solution respectively. In order to obtain spinning solution,we rotate spacetime following Newman-Janis algorithm for Kerr and Kerr-Newman. We discussed also the main feture of Kerr-Newman near event horizon.

**Keywords**: Einstein Field Equation, Singularity, Kerr-Newman

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**"KAJIAN RUANG-WAKTU KERR-NEWMAN DALAM GRAVITASI EINSTEIN"**

Tugas akhir ini diharapkan dapat membantu rekan-rekan mahasiswa S1 yang ingin belajar lebih mendalam tentang Fisika terutama di sekitar topik solusi persamaan medan Einstein.

Terselesaikannya tugas akhir ini tidak luput dari bantuan berbagai pihak. Penulis mengucapkan terima kasih dengan setulus hati kepada:

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Surabaya, 31 Januari 2016

Penulis

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# DAFTAR GAMBAR

# DAFTAR TABEL

# BAB 1

# PENDAHULUAN

## 1.1 Latar Belakang

Gravitasi antar berbagai partikel telah diterangkan dengan sangat baik oleh Newton dengan Hukum Gravitasi Universal-nya yaitu partikel-partikel di alam semesta saling tarik menarik dengan gaya yang berbanding lurus dengan hasil kali dari massa masing-maisng partikel dan berbanding terbalik dengan kuadrat dari jarak diantara mereka. Hukum ini sangat berhasil dalam menerangkan sifat-sifat pergerakan benda langit dengan ketelitian yang cukup tinggi.

Pada tahun 1905, Albert Einstein mencetuskan teorinya yang diberi nama Teori Relativitas Khusus. Teori ini merumuskan ulang bagaimana formasi ruang-waktu. Teori ini didasarkan pada 2 postulat yaitu semua hukum fisika harus berbentuk sama (invarian) pada semua kerangka acuan dan kecepatan cahaya adalah kecepatan mutlak. Koreksi dari hukum gravitasi universal Newton terhadap teori ini adalah jika misal terdapat partikel-partikel yang salah satunya digerakkan dan yang lain diam (relatif terhadap partikel yang lain), maka gaya gravitasi antar partikel tersebut akan berubah secara spontan tak peduli berapapun besarnya. Kata ”spontan” dipakai karena perubahan jarak antara partikel 1 dengan partikel lainnya akan berubah saat itu juga. Hal ini berarti komunikasi antar berbagai partikel tersebut terjadi sangat cepat (spontan) yang melebihi kecepatan cahaya tak bergantung berapapun besar jaraknya, sehingga hal ini telah melanggar postulat ke-2 Teori Relativitas Khusus.

Teori Relativitas Khusus dapat menjelaskan dengan sangat baik untuk kerangka acuan yang satu yang bergerak relatif dengan kerangka acuan yang lain dengan kecepatan konstan yang sangat tinggi (mendekati kecepatan cahaya). Untuk kerangka acuan yang bergerak relatif dengan percepatan konstan, maka pada tahun 1916 Albert Einstein men-generalisasi Teori Relativitas Khusus dengan mencetuskan Teori Relativitas Umum yaitu teori yang menjelaskan kerangka acuan yang saling bergerak relatif dengan suatu percepatan. Dalam teori ini gravitasi bukan dipandang sebagai gaya, tetapi lebih sebagai manifestasi dari kelengkungan ruang-waktu. Efek kelengkungan ruang-waktu terjadi jika ada benda bermassa. Semakin besar massa suatu benda, maka semakin besar pula efek kelengkungan ruang-waktunya. Formulasi dari kelengkungan ruang-waktu ini dirumuskan dalam persamaannya yang terkenal yaitu Persamaan Medan Einstein.

Solusi dari persamaan medan Einstein pertama kali dikerjakan oleh Karl Schwarzschild pada tahun yang sama sejak Teori Relativitas Umum dipublikasikan. Schwarzschild memberikan solusi statik dan simetri bola untuk persamaan medan Einstein. Hasil yang didapatkan adalah lubang hitam (black hole) statik. Lubang hitam adalah suatu benda dengan kerapatan massa sangat tinggi sehingga kelengkungan ruang-waktu di sekitarnya sangat kuat. Kekuatan dari kelengkungan ruang-waktu ini mampu menarik benda-benda di sekitarnya hingga cahaya pun tidak bisa keluar setelah masuk ke dalamnya.

Secara astrofisika, benda langit umumnya berputar (berotasi). Karena solusi Schwarzschild hanya untuk lubang hitam yang tidak berotasi, maka seharusnya terdapat solusi lain untuk sumber massa yang berotasi. Pada tahun 1963, Roy Patrick Kerr menemukan solusi bagi persamaan medan Einstein untuk sumber massa yang berotasi.

Jika persamaan medan Einstein digabungkan dengan persamaan Maxwell, maka akan didapat persamaan medan Einstein-Maxwell. Solusi statik untuk persamaan ini dikerjakan oleh Hans Jacob Reissner dan Gunnar Nordstrom yang diberi nama solusi Reissner-Nordstrom yaitu solusi untuk lubang hitam yang statik, bermuatan dan simetri bola. Kemudian jika lubang hitam yang bermuatan ini juga berotasi, maka akan didapatkan solusi Kerr-Newman.

## 1.2 Perumusan Masalah

Rumusan masalah pada Tugas Akhir ini adalah bagaimana mendapatkan solusi Kerr dan Kerr-Newman dengan menerapkan Algoritma Newman-Janis pada metrik Schwarzschild dan Reissner-Nordstrom yang diungkapkan dalam koordinat null.

## 1.3 Tujuan

Tujuan yang ingin dicapai pada Tugas Akhir ini adalah untuk mendapatkan solusi Kerr dan Kerr-Newman dengan menerapkan Algoritma Newman-Janis pada metrik Schwarzschild dan Reissner-Nordstrom yang diungkapkan dalam koordinat null. Kemudian dicari sifat-sifat didekat horison peristiwa dari metrik Kerr-Newman

## 1.4 Metode Penelitian

Metode yang digunakan dalam penyusunan Tugas Akhir ini adalah metode analitis dari studi literatur.

## 1.5 Sistematika Penulisan

Dalam penulisan Tugas Akhir ini , terdiri dari 6 bab. Pada bab I diuraikan mengenai motivasi dan sejarah singkat munculnya Teori Relativitas Umum. Pada bab II akan diuraikan beberapa Persamaan Medan Einstein yang dipakai yaitu bagaimana mendapat Persamaan Medan Einstein serta Algoritma Newman-Janis yaitu metode untuk mendapatkan solusi lubang hitam berotasi dari lubang hitam statik. Pada bab III akan diturunkan solusi statik dari Persamaan Medan Einstein yaitu solusi Schwarzschild dan solusi Reissner-Nordstrom. Pada bab IV akan diturunkan solusi berotasi Persamaan Medan Einstein yaitu solusi Kerr dan Solusi Kerr-Newman dengan menggunakan Algoritma Newman-Janis. Bab V adalah diskusi yang berisi penjelasan mengenasi sifat-sifat di sekitar horison peristiwa lubang hitam Kerr-Newman. Bab 6 adalah kesimpulan dan saran. Lampiran berisi penurunan rumus secara detail. Pada tugas akhir ini, indeks yang digunakan untuk huruf yunani (misal $\mu$) berjalan dari (0,1,2,3) dan untuk indeks huruf latin (misal $i$) untuk indeks yang berjalan dari (1,2,3), serta tanda yang dipilih untuk metrik adalah -+++. Untuk menjaga kerapian penulisan, penurunan rumus pada tiap bab dituliskan sesingkat dan seringkas mungkin, sedangkan penurunan secara detailnya dapat dilihat pada Lampiran.

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# BAB 2

# PERSAMAAN MEDAN EINSTEIN

Persamaan medan Einstein adalah persamaan yang menggambarkan kelengkungan ruang waktu yang diakibatkan oleh distribusi materi di sekitarnya. Penjelasan mengenai persamaan medan Einstein beserta derivasinya ditunjukan pada subbab-subbab berikut.

## 2.1 Persamaan Geodesik

Geodesik adalah jarak terpendek antara 2 titik. Pada ruang koordinat kartesian, geodesiknya adalah garis lurus, sedangkan pada koordinat bola geodesiknya adalah panjang busur terpendek yang menghubungkan 2 titik tersebut pada lingkaran bola dengan jari-jari terbesar. Perhatikan gambar berikut,


**Gambar 2.1:** Geodesik pada ruang datar 2 dimensi

Dari Gambar 2.1 diatas, terlihat bahwa jarak terdekat antara titik penalti bola ke titik tengah lapangan adalah garis lurus PQ. Sedangkan dari Gambar 2.2 di bawah ini, terlihat bahwa jarak terdekat antara 2 titik pada ruang lengkung 2 dimensi adalah kurva lengkung PQ.

Misal terdapat 2 titik A dan B yang dihubungkan oleh 3 buah lintasan seperti Gambar 2.3 di bawah ini. Koordinatnya diberikan oleh persamaan $x^\mu = x^\mu(s_A)$ dan $x^\mu = x^\mu(s_B)$, dengan $s$ adalah suatu parameter. Dari


**Gambar 2.2:** Geodesik pada ruang lengkung 2 dimensi

gambar di atas, $C'$ dan $C''$ bukanlah panjang stasioner tetapi merupakan suatu deviasi/simpangan dari $C$. Untuk memperoleh panjang stasioner ini, maka variasi $\delta x^\mu$ pada 2 titik $A$ dan $B$ harus bernilai nol.

$$\delta x^\mu(s_A) = \delta x^\mu(s_B) = 0 \tag{2.1}$$

Kemudian didefinisikan Lagrangian adalah fungsi dari $x^\mu$ dan turunanannya terhadap parameter $s$

$$\mathcal{L} = \mathcal{L}\left(x^\mu, \frac{dx^\mu}{ds}, s\right) \tag{2.2}$$

sehingga integral aksi adalah

$$I = \int_{s_A}^{s_B} \mathcal{L}\left(x^\mu, \frac{dx^\mu}{ds}, s\right) ds \tag{2.3}$$

Agar didapatkan panjang kurva stasioner, maka variasi dari integral di atas harus bernilai nol

$$\delta I = \int_{s_A}^{s_B} \delta\mathcal{L}\left(x^\mu, \frac{dx^\mu}{ds}, s\right) ds = 0 \tag{2.4}$$

dengan definisi variasi adalah $\delta\mathcal{L}\left(x^\mu, \frac{dx^\mu}{ds}, s\right)$

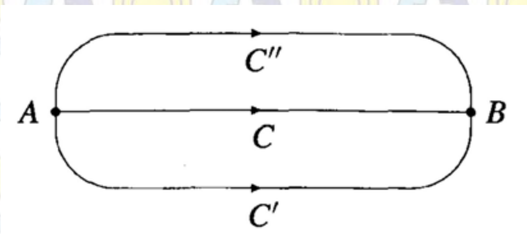
$$\delta\mathcal{L} = \mathcal{L}\left(x^\mu + \delta x^\mu, \frac{dx^\mu}{ds} + \delta\frac{dx^\mu}{ds}, s\right) - \mathcal{L}\left(x^\mu, \frac{dx^\mu}{ds}, s\right) \tag{2.5}$$

Suku pertama pada pers.(2.5) diekspansikan dalam deret Taylor (lihat Lampiran A.1) kemudian disubstitusikan ke pers.(2.4) sehingga

$$\delta I = \int_{s_A}^{s_B} \left[\frac{d}{ds}\left(\frac{\partial\mathcal{L}}{d\left(\frac{\partial x^\mu}{\partial s}\right)}\right) - \frac{\partial\mathcal{L}}{\partial x^\mu}\right] \delta x^\mu ds = 0 \tag{2.6}$$

sehingga didapatkan persamaan Lagrange

$$\frac{d}{ds}\left(\frac{\partial\mathcal{L}}{d\left(\frac{\partial x^\mu}{\partial s}\right)}\right) - \frac{\partial\mathcal{L}}{\partial x^\mu} = 0 \tag{2.7}$$




**Gambar 2.3:** 2 titik yang dihubungkan dengan 3 lintasan

Panjang stasioner untuk suatu kurva adalah

$$I = \int dl = \int_{s_A}^{s_B} \left( g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \right)^{\frac{1}{2}} ds \tag{2.8}$$

dengan $s$ adalah suatu parameter. Bentuk integral tersebut didapatkan dari persamaan metrik berikut:

$$dl^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{2.9}$$

sehingga Lagrangian yang bersesuaian dengan bentuk bentuk integral (2.8) diatas adalah

$$\mathcal{L} = \left( g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \right)^{\frac{1}{2}} \tag{2.10}$$

Suku pertama pers.(2.7) di atas dapat diperoleh dari pers.(2.10) yaitu

$$\frac{d}{ds} \left[ \frac{\partial \mathcal{L}}{\partial \left( \frac{dx^\lambda}{ds} \right)} \right] = g_{\mu\lambda} \frac{d^2 x^\mu}{ds^2} + \frac{1}{2} \left\{ \frac{\partial g_{\mu\lambda}}{\partial x^\nu} + \frac{\partial g_{\nu\lambda}}{\partial x^\mu} \right\} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \tag{2.11}$$

dan suku ke-2 pers.(2.7)

$$\frac{\partial \mathcal{L}}{\partial x^\lambda} = \frac{1}{2} \left( \frac{\partial g_{\mu\nu}}{\partial x^\lambda} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \right) \tag{2.12}$$

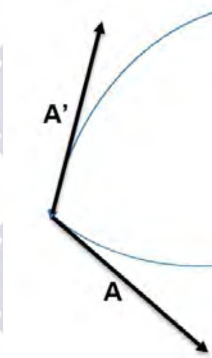
kemudian disubstitusikan pada pers.(2.7) di atas. Hasil yang diperoleh adalah persamaan geodesik sebagai berikut.

$$\frac{d^2 x^\rho}{ds^2} + \Gamma^\rho_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0 \tag{2.13}$$

Pada suku ke-2 pers.(2.13) di atas, terdapat koefisien $\Gamma^\rho_{\mu\nu}$ yang disebut sebagai koefisien koneksi atau simbol Christoffel yang merupakan turunan pertama dari tensor metrik. Pada ruang-waktu datar (Minkowski), metrik akan bernilai konstan yang menyebabkan simbol Christoffel bernilai nol sehingga dari pers.(2.13) di atas, akan diperoleh geodesik berupa garis lurus.

## 2.2 Tensor kurvatur

Jika terdapat suatu vektor $A_\mu$ pada ruang Minkowski dan ditranslasi ditranslasikan secara paralel pada dua titik sepanjang lintasan tertutup C, maka posisi vektor tersebut akan berimpit dengan vektor awal sebelum ditranslasi. Tetapi pada ruang Riemann, jika suatu vektor ditranslasikan secara paralel pada dua titik sepanjang lintasan tertutup C, maka operasinya akan bergantung pada lintasannya seperti pada Gambar fig:2.4 dibawah ini.




**Gambar 2.4:** Translasi vektor pada ruang lengkung

Jika perubahan vektor itu adalah $\Delta A_\mu$, maka:

$$\Delta A_\mu = \oint_C \delta A_\mu = \oint_C \Gamma^\nu_{\mu\rho} A_\nu dx^\rho \tag{2.14}$$

Dari teorema Stokes, integral kontur tertutup $C$ dapat ditransformasi ke integral luasan $S$ sehingga

$$\Delta A_\mu = \frac{1}{2} \oint_C \left[ D_\gamma \left( \Gamma^\rho_{\mu\nu} A_\rho \right) - D_\nu \left( \Gamma^\rho_{\mu\gamma} A_\rho \right) \right] dS^{\gamma\nu} \tag{2.15}$$

dengan $D_\gamma$ adalah operator turunan kovarian. Operator ini dapat disimplifikasi menjadi operator turunan biasa non-tensor $D_\gamma \to \partial_\gamma$. Integral (2.15) akan menjadi

$$\begin{aligned}\Delta A_\mu &= \frac{1}{2}\oint_C \left[\partial_\gamma\left(\Gamma^\rho_{\mu\nu}A_\rho\right) - \partial_\nu\left(\Gamma^\rho_{\mu\gamma}A_\rho\right)\right] dS^{\gamma\nu} \\ &= \frac{1}{2}\oint_C [\partial_\gamma\Gamma^\rho_{\mu\nu} - \partial_\nu\Gamma^\rho_{\mu\gamma} \\ &\quad +\Gamma^\lambda_{\mu\nu}\Gamma^\rho_{\lambda\gamma} - \Gamma^\lambda_{\mu\gamma}\Gamma^\rho_{\lambda\nu}]A_\rho dS^{\gamma\lambda} \\ &\equiv \frac{1}{2}\oint_C R^\rho_{\mu\gamma\nu}A_\rho dS^{\gamma\lambda} \qquad (2.16)\end{aligned}$$

dengan $R^\rho_{\mu\gamma\nu}$ adalah tensor kurvatur Riemann

$$R^\rho_{\mu\gamma\nu} = \partial_\gamma\Gamma^\rho_{\mu\nu} - \partial_\nu\Gamma^\rho_{\mu\gamma} + \Gamma^\lambda_{\mu\nu}\Gamma^\rho_{\lambda\gamma} - \Gamma^\lambda_{\mu\gamma}\Gamma^\rho_{\lambda\nu} \qquad (2.17)$$

Tensor kurvatur Riemann menggambarkan bagaimana bentuk dari kelengkungan ruang-waktu. Kemudian didefinisikan tensor Ricci yang merupakan tensor rank-2 sebagai berikut

$$R_{\mu\gamma} \equiv R^\rho_{\mu\gamma\rho} = \partial_\gamma\Gamma^\rho_{\mu\rho} - \partial_\rho\Gamma^\rho_{\mu\gamma} + \Gamma^\lambda_{\mu\rho}\Gamma^\rho_{\lambda\gamma} - \Gamma^\lambda_{\mu\gamma}\Gamma^\rho_{\lambda\rho} \qquad (2.18)$$

serta scalar Ricci adalah

$$R \equiv R^\mu_\mu = g^{\mu\gamma}R_{\mu\gamma} \qquad (2.19)$$

## 2.3 Tensor Energi-Momentum

Tensor Energi-Momentum adalah kuantitas tensor yang mendekripsikan kerapatan dan fluks dari energi dan momentum. Ditinjau aksi yang berbentuk

$$I = \int \mathcal{L}\left(q, \frac{\partial q}{\partial x^\mu}\right) dV\, dt = \frac{1}{c}\int \mathcal{L}\left(q, \frac{\partial q}{\partial x^\mu}\right) d\Omega \qquad (2.20)$$

dengan $\mathcal{L}$ adalah fungsi dari kuantitas $q$ yang mendeskripsikan keadaan sistem dan turunan pertamanya. Karena $\int \mathcal{L} dV$ adalah suatu Lagrangian, maka $\int \mathcal{L}$ adalah rapat Lagrangian. Penulisan $\frac{\partial q}{\partial x^\mu}$ bisa ditulis sebagai $\partial_\mu q = q_{,\mu}$. Persamaan gerak bisa didapatkan dengan memberikan variasi pada pers.(2.20) di atas sehingga

$$\begin{aligned}\delta I &= \frac{1}{c}\int \delta\mathcal{L}\,(q, q_{,\mu})d\Omega \\ &= \frac{1}{c}\int \left(\frac{\partial\mathcal{L}}{\partial q}\delta q + \frac{\partial\mathcal{L}}{\partial(q_{,\mu})}\delta(q_{,\mu})\right) d\Omega \end{aligned} \tag{2.21}$$

Suku ke-2 persamaan di atas dapat dijabarkan menjadi

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial(q_{,\mu})}\delta(q_{,\mu}) &= \frac{\partial\mathcal{L}}{\partial(q_{,\mu})}\delta\frac{\partial q}{\partial x^\mu} \\ &= \frac{\partial\mathcal{L}}{\partial q_{,\mu}}\frac{\partial}{\partial x^\mu}(\delta q) \\ &= \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial q_{,\mu}}\delta q\right) - \delta q\frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial q_{,\mu}}\right)\end{aligned} \tag{2.22}$$

sehingga pers.(2.21) menjadi

$$\delta I = \frac{1}{c}\int \left(\frac{\partial\mathcal{L}}{\partial q}\delta q + \frac{\partial}{\partial(x^\mu)}\left(\frac{\partial\mathcal{L}}{\partial q_{,\mu}}\delta q\right) - \delta q\frac{\partial}{\partial x^\mu}\frac{\partial\mathcal{L}}{\partial q_{,\mu}}\right) d\Omega \tag{2.23}$$

Dari teorema Gauss $\int_\Omega \partial_\mu F^\mu = \int_S F^\mu n_\mu dS = 0$ dengan $F^\mu$ adalah suatu medan tensor, maka suku ke-2 persamaan di atas lenyap (lihat lampiran A.3) sehingga:

$$\begin{aligned}\delta I &= \frac{1}{c}\int \left(\frac{\partial\mathcal{L}}{\partial q}\delta q - \delta q\frac{\partial}{\partial x^\mu}\frac{\partial\mathcal{L}}{\partial q_{,\mu}}\right) d\Omega \\ 0 &= \frac{1}{c}\int \left(\frac{\partial\mathcal{L}}{\partial q} - \frac{\partial}{\partial x^\mu}\frac{\partial\mathcal{L}}{\partial q_{,\mu}}\right)\delta q d\Omega\end{aligned} \tag{2.24}$$

Sehingga didapatkan

$$\frac{\partial}{\partial x^\mu}\frac{\partial\mathcal{L}}{\partial q_{,\mu}} - \frac{\partial\mathcal{L}}{\partial q} = 0 \tag{2.25}$$

yang disebut sebagai persamaan Euler-Lagrange. Karena rapat lagrangian merupakan fungsi dari kuantitas $q$ dan turunan pertamanya $q_{,\mu}$ yaitu $\mathcal{L} = \mathcal{L}(q, q_{,\mu})$, maka

$$\frac{\partial \mathcal{L}}{\partial x^\mu} = \frac{\partial \mathcal{L}}{\partial q}\frac{\partial q}{\partial x^\mu} + \frac{\partial \mathcal{L}}{\partial q_{,\nu}}\frac{\partial q_{,\nu}}{\partial x^\mu} \tag{2.26}$$

Kemudian substitusikan nilai dari $\frac{\partial \mathcal{L}}{\partial q}$ dari pers.(2.25) ke dalam pers.(2.26) di atas sehingga:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x^\mu} &= \frac{\partial}{\partial x^\nu}\frac{\partial \mathcal{L}}{\partial q_{,\nu}}\frac{\partial q}{\partial x^\mu} + \frac{\partial \mathcal{L}}{\partial q_{,\mu}}\frac{\partial q_{,\mu}}{\partial x^\nu} \\ &= \frac{\partial}{\partial x^\nu}\left(q_{,\mu}\frac{\partial \mathcal{L}}{\partial q_{,\nu}}\right)\end{aligned} \tag{2.27}$$

dengan $\frac{\partial \mathcal{L}}{\partial x^\mu}$ bisa dituliskan sebagai $\frac{\partial \mathcal{L}}{\partial x^\mu} = \delta^\nu_\mu \frac{\partial \mathcal{L}}{\partial x^\nu}$, sehingga pers.(2.27) menjadi

$$\delta^\nu_\mu \frac{\partial \mathcal{L}}{\partial x^\nu} = \frac{\partial}{\partial x^\nu}\left(q_{,\mu}\frac{\partial \mathcal{L}}{\partial q_{,\mu}}\right) \tag{2.28}$$

dan didapatkan

$$\frac{\partial}{\partial x^\nu}\left(q_{,\mu}\frac{\partial \mathcal{L}}{\partial q_{,\mu}} - \delta^\nu_\mu \mathcal{L}\right) \equiv \frac{\partial}{\partial x^\mu} T^\nu_\mu = 0 \tag{2.29}$$

dengan $T^\nu_\mu$ adalah definisi untuk tensor campuran energi-momentum.

Untuk medan elektromagnetik, Lagrangian medan elektromagnetik bebas diberikan oleh:

$$\mathcal{L}_F(F_{\mu\nu}) = -\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} \tag{2.30}$$

dengan asumsi bahwa tidak ada sumber arus ($J^\nu = 0$) sehinga persamaan Maxwell menjadi

$$\partial_\mu F^{\mu\nu} = 0 \tag{2.31}$$

dengan $F^{\mu\nu}$ merupakan tensor kuat medan. Tensor kuat medan $F^{\mu\nu}$ ini didefinisikan dari potensial vektor $A^\mu$ sebagai berikut.

$$F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu} \tag{2.32}$$

dengan bentuk kovariannya adalah

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \tag{2.33}$$

Untuk mendefinisikan tensor energi momentum pada medan Elektromagnetik, maka dihitung dulu turunan dari rapat lagrangian sebagai berikut:

$$\partial_{\nu}\mathcal{L} = \frac{\partial\mathcal{L}}{\partial A_{\gamma}}\partial_{\mu}A_{\gamma} + \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}A_{\gamma})}\partial_{\nu}(\partial_{\mu}A_{\gamma}) \tag{2.34}$$

dengan

$$\frac{\partial\mathcal{L}}{\partial A_{\gamma}}\partial_{\mu}A_{\gamma} = \partial_{\mu}\left[\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}A_{\gamma})}\right]\partial_{\mu}A_{\gamma} \tag{2.35}$$

serta dari definisi

$$\partial_{\nu}\mathcal{L} = \delta^{\mu}_{\nu}\partial_{\mu}\mathcal{L} \tag{2.36}$$

maka didapatkan

$$\partial_{\mu}\left[\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}A_{\gamma})}\partial_{\mu}A_{\gamma} - \delta^{\mu}_{\nu}\mathcal{L}\right] \equiv \partial_{\mu}T^{\mu}_{\nu} = 0 \tag{2.37}$$

dengan $T^{\mu}_{\nu}$ adalah tensor energi-momentum campuran. Dengan mensubstitusikan pers.(2.30) ke dalam persamaan di atas, maka didapatkan

$$T^{\mu}_{\nu} = -\frac{1}{\mu_0}F^{\mu\gamma}F_{\nu\gamma} + \frac{1}{4\mu_0}\delta^{\mu}_{\nu}F^{\gamma\lambda}F_{\gamma\lambda} \tag{2.38}$$

dengan bentuk kontravarian dan kovariannya masing-masing adalah:

$$T^{\mu\nu} = -\frac{1}{\mu_0}F^{\nu\gamma}F^{\mu}\gamma + \frac{1}{4\mu_0}g^{\mu\nu}F^{\gamma\lambda}F_{\gamma\lambda} \tag{2.39}$$

$$T_{\mu\nu} = -\frac{1}{\mu_0} F_\nu^\gamma F_{\mu\gamma} + \frac{1}{4\mu_0} g_{\mu\nu} F^{\gamma\lambda} F_{\gamma\lambda} \tag{2.40}$$

Tensor energi momentum pers.(2.40) menggambarkan distribusi energi dan momentum yang disebabkan oleh medan elektromagnetik

## 2.4 Persamaan Medan Einstein

Persamaan medan Einstein adalah persamaan yang menggambarkan kelengkungan ruang waktu yang diakibatkan oleh distribusi materi di sekitarnya. Bentuk kelengkungan ruang-waktu digambarkan oleh bentuk tensor metrik dan bentuk tensor metrik dipengaruhi oleh bagaiamana distribusi massa sebagai sumber dari gravitasi. Integral aksi total yang disebabkan aksi massa sumber dan aksi oleh gravitasi adalah:

$$I = I_G + I_M \tag{2.41}$$

dengan $I_G$ merupakan aksi oleh medan gravitasi pada ruang vakum (dimana tidak ada sumber medan) dan $I_M$ merupakan aksi oleh massa sumber. Kemudian diambil variasi dari pers.(2.41) di atas agar diperoleh aksi minimum yaitu

$$\begin{aligned} \delta I &= \delta I_G + \delta I_M \\ 0 &= \delta I_G + \delta I_M \\ -\delta I_M &= \delta I_G \end{aligned} \tag{2.42}$$

dengan nilai dari masing-masing $\delta I_G$ dan $\delta I_M$ dapat dilihat pada lampiran A.4. Jika kedua nilai ini disubstitusikan pada pers.(2.42), maka didapatkan

$$\begin{aligned} -\delta I_M &= \delta I_G \\ -\frac{1}{2c}\int_\Omega T_{\mu\nu}\delta g^{\mu\nu}\sqrt{-g}d\Omega &= -\frac{c^3}{16\pi G}\int_\Omega \left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\sqrt{-g}\delta g^{\mu\nu}d\Omega \end{aligned} \tag{2.43}$$

dari pers.(2.43) di atas, maka akan di dapatkan:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{2.44}$$

Pers.(2.44) merupakan Persamaan Medan Einstein. Ruas kiri persamaan medan Einstein menggambarkan kelengkungan ruang-waktu dan ruas kanannya menggambarkan distribusi materi. Interpretasi dari persamaan ini adalah materi menyebabkan ruang waktu melengkung atau kelengkungan ruang waktu memerintahkan materi untuk bergerak.

Jika pers.(2.44) dibentuk dalam tensor campuran $R^{\mu}_{\nu}$ dan $T^{\mu}_{\nu}$ maka akan didapatkan:

$$R^{\mu}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu}R = \frac{8\pi G}{c^4}T^{\mu}_{\nu} \tag{2.45}$$

Kemudian dilakukan kontraksi indeks $\nu \to \mu$ pada persamaan di atas sehingga

$$R^{\mu}_{\mu} - \frac{1}{2}\delta^{\mu}_{\mu}R = \frac{8\pi G}{c^4}T^{\mu}_{\mu} \tag{2.46}$$

Karena

$$\delta^{\mu}_{\mu} = \delta^{0}_{0} + \delta^{1}_{1} + \delta^{2}_{2} + \delta^{3}_{3} = 4 \tag{2.47}$$

maka

$$\begin{aligned} R - 2R &= \frac{8\pi G}{c^4}T \\ R &= -\frac{8\pi G}{c^4}T \end{aligned} \tag{2.48}$$

sehingga pers.(2.44) dapat diungkapkan dengan bentuk yang lain menjadi:

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) \tag{2.49}$$

## 2.5 Algoritma Newman-Janis

Schwarzschild telah memecahkan solusi untuk Persamaan Medan Einstein di atas yaitu solusi untuk medan statik dan simetri bola. Penjelasan tentang solusi Schwarzschild ini disajikan dalam bab 3. Karena umumnya benda langit berotasi, maka dari solusi Schwarzschild ini kemudian dikembangkan solusi untuk yang non-statik yakni berotasi yaitu solusi Kerr. Setelah ditemukannya solusi Kerr yang masih murni, Newman dan Janis dapat menunjukkan bahwa solusi Kerr dapat diturunkan dengan cara lain yang selanjutnya disebut sebagai Algoritma Newman-Janis. Algoritma ini diterapkan pada metrik Schwarzschild yang kemudian menghasilkan solusi Kerr dan jika Algoritma ini diterapkan pada metrik Reissner-Nordstrom (solusi statik dan simetri bola dari persamaan medan Einstein-Maxwell), maka akan didapatkan solusi Kerr-Newman. Algoritma Newman-Janis adalah sebagai berikut:

1. Menuliskan bentuk metrik ke dalam koordinat null dengan transformasi sebagai berikut:

   $$u = t - r^* \text{dengan } r^* \text{ merupakan tortoise koordinat}$$

2. Menuliskan tensor metrik kontravarian dari metrik pada metode pertama dalam suku-suku vektor-4 null yaitu:

   $$g^{\mu\nu} = -l^\mu n^\nu - l^\nu n^\mu + m^\mu \bar{m}^\nu + \bar{m}^\mu m^\nu$$

   dengan

   $$\begin{aligned} l_\mu l^\mu = n_\mu n^\mu &= m_\mu m^\mu = 0 \\ -l_\mu n^\mu &= m_\mu \bar{m}^\mu = 1 \\ l_\mu m^\mu &= n_\mu m^\mu = 0 \end{aligned}$$

3. Mentransformasikan koordinat $x^\mu$ menjadi koordinat kompleks $\tilde{x}^\mu$

   $$x^\mu \rightarrow \tilde{x}^\mu \tag{2.50}$$

4. Dilakukan transformasi sebagai berikut

   $$\tilde{x}^\mu = x^\mu + ia\, cos\, x^2(\delta_0^\mu - \delta^\mu_{\delta^\mu_1}) \tag{2.51}$$

Metrik yang dihasilkan dari algoritma ini adalah metrik Kerr dan metrik Kerr-Newman yang dinyatakan dalam koordinat null. Kemudian untuk menganalisa sifat-sifat seperti singularitas dan event horizon, metrik ini ditransformasikan dalam koordinat *Boyer-Lidquist* yaitu dengan cara:

$$dt = du + \frac{r^2 + a^2}{\Delta} dr, \quad d\psi = d\phi + \frac{a}{\Delta} dr \tag{2.52}$$

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# BAB 3

# SOLUSI STATIK PERSAMAAN MEDAN EINSTEIN

## 3.1 Solusi Schwarzchild

Solusi Schwazschild adalah solusi untuk Persamaan Medan Einstein untuk kasus statik dan simetri bola dari massa $M$. Statik berarti tensor metrik $g_{\mu\nu}$ tidak bergantung waktu atau $\partial_\gamma g_{\mu\nu} = 0$, serta $ds^2$ harus invarian terhadap transformasi $x^0 \rightarrow -x^0$ (pembalikan waktu). Konsekuensinya adalah $ds^2$ tidak boleh mengandung suku $dx^i\, dx^0$yang menjadikannya tidak invarian terhadap transformasi pembalikan waktu atau $g_{i0} = g_{0i} = 0$. Karena tensor metrik tidak bergantung waktu serta elemen garisnya tidak boleh terdapat suku silang antara $t$ dengan $(r, \theta, \phi)$, maka tensor metrik $g_{\mu\nu}$ hanya memiliki elemen diagonal yang koefisiennya merupakan fungsi dari parameter $r$ saja untuk mempertahankan bentuk simetri bola.

$$ds^2 = -U(r)dt^2 + V(r)dr^2 + W(r)r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.1}$$

Dari kondisi khusus yang didapatkan di sini, maka 10 komponen bebas tensor metrik yang secara umum bergantung pada $x^\mu$ menjadi tereduksi hanya menjadi 4 komponen bebas yang merupakan fungsi $r$ saja. Karena persamaan medan merupakan turunan ke-2 dari tensor metrik, maka 3 fungsi $U(r), V(r) dan W(r)$ di atas bisa direduksi hanya menjadi 2 fungsi saja. Karena $r$ merupakan parameter radial, maka bisa digantikan dengan sembarang fungsi $r$. Misal kan diambil $Wr^2 = \hat{r}^2$, maka $\hat{r} = \sqrt{W}r$, serta

$$\frac{d\hat{r}}{dr} = \sqrt{W}\left(1 + \frac{r}{2W}\frac{dW}{dr}\right) \tag{3.2}$$

maka

$$V\, dr^2 = \frac{V}{W}\left(1 + \frac{r}{2W}\frac{dW}{dr}\right)^{-2} d\hat{r}^2 \equiv \hat{V} d\hat{r}^2 \tag{3.3}$$

dengan $V \equiv \hat{V}$. Dengan cara yang sama maka bisa didapatkan $U \equiv \hat{U}$. Dengan mengganti $r$ menjadi $\hat{r}$, maka elemen garis diatas akan menjadi

$$ds^2 = -\hat{V}(\hat{r})dt^2 + \hat{U}(\hat{r})dr^2 + \hat{r}^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.4}$$

Kemudian dengan menghilangkan tanda topi pada persamaan di atas serta dipilih fungsi dari $U$ dan $V$ adalah

$$U(r) = e^{2\nu(r)}, \text{dan } V(r) = e^{2\lambda(r)} \tag{3.5}$$

maka elemen garisnya akan menjadi

$$ds^2 = -e^{2\nu}c^2dt^2 + e^{2\lambda}dr^2 + (r^2d\theta^2 + r^2\sin^2\theta d\phi^2) \tag{3.6}$$

Dari elemen garis pers.(3.6) di atas, maka didapatkan tensor metrik $g_{\mu\nu}$ sebagai berikut

$$g_{\mu\nu} = \begin{pmatrix} -e^{2\nu} & 0 & 0 & 0 \\ 0 & e^{2\lambda} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{3.7}$$

dengan bentuk kontravarian dari tensor metrik diatas adalah

$$g^{\mu\nu} = \begin{pmatrix} -e^{-2\nu} & 0 & 0 & 0 \\ 0 & e^{-2}\lambda & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^{-2}\sin^{-2}\theta \end{pmatrix} \tag{3.8}$$

Nilai dari simbol Christoffel jenis ke-2 dari tensor metrik pers.(3.7) di atas adalah

$$\Gamma^{\rho}_{\mu\nu} = g^{\rho\sigma}\Gamma_{\sigma,\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left(\partial_\nu g_{\mu\sigma} + \partial_\mu g_{\nu\sigma} - \partial_\sigma g_{\mu\nu}\right)$$

yang berjumlah sebanyak 64 komponen. Komponen-komponen yang tidak bernilai 0 adalah:

$$
\begin{aligned}
\Gamma^0_{01} &= \Gamma^0_{10} = \nu' \\
\Gamma^1_{00} &= \nu' e^{(2\nu-2\lambda)} \\
\Gamma^1_{11} &= \lambda' \\
\Gamma^1_{22} &= -re^{-2\lambda} \\
\Gamma^1_{33} &= -r\sin^2\theta\, e^{-2\lambda} \\
\Gamma^2_{12} &= \Gamma^2_{21} = \frac{1}{r} \\
\Gamma^2_{33} &= -\sin\theta\cos\theta \\
\Gamma^3_{13} &= \Gamma^3_{31} = \frac{1}{r} \\
\Gamma^3_{23} &= \Gamma^3_{32} = \cot\theta
\end{aligned}
\tag{3.9}
$$

Tensor Ricci dapat dihitung dengan menggunakan persamaan

$$R_{\mu\nu} = \partial_\nu \Gamma^\sigma_{\mu\sigma} - \partial_\sigma \Gamma^\sigma_{\mu\nu} + \Gamma^\rho \mu_\nu \Gamma^\sigma_{\rho\nu} - \Gamma^\rho_{\mu\nu}\Gamma^\sigma_{\rho\sigma} \tag{3.10}$$

Karena Tensor Ricci bersifat simetri ($R_{\mu\nu} = R_{\nu\mu}$), maka hanya memiliki 10 komponen bebas. Untuk komponen R$_{i0}, (i = 1, 2, 3)$ :

$$R_{i0} = \partial_0 \Gamma^\sigma_{i\sigma} - \partial_\sigma \Gamma^\sigma_{i0} + \Gamma^\rho_{i\sigma}\Gamma^\sigma_{\rho 0} - \Gamma^\rho_{i0}\Gamma^\sigma_{\rho\sigma} \tag{3.11}$$

dengan kondisi statik mensyaratkan bahwa $\partial_0 g_{\mu\nu} = 0$ sehingga $\partial_0 \Gamma^\sigma_{i\sigma} = 0$. Tensor Ricci menjadi

$$R_{i0} = -\partial_j \Gamma^j_{i0} + \Gamma^\rho_{ij}\Gamma^j_{\rho 0} - \Gamma^\rho_{i0}\Gamma^j_{\rho j} \tag{3.12}$$

Dengan menggunakan nilai $\Gamma^i_{j0} = 0$, $\Gamma^\rho_{0\rho} = 0$ dan $\Gamma^0_{ij} = 0$, maka didapatkan

$$R_{i0} = R_{0i} = 0 \tag{3.13}$$

sehingga komponen tensor Ricci yang tersisa adalah komponen dalam arah diagonal ($R_{\mu\mu}$). Nilai dari $R_{\mu\mu}$ ini adalah

$$R_{\mu\mu} = \partial_\mu \Gamma^\sigma_{\mu\sigma} - \partial_\sigma \Gamma^\sigma_{\mu\mu} + \Gamma^\sigma_{\rho\mu}\Gamma^\rho_{\mu\sigma} - \Gamma^\rho_{\mu\mu}\Gamma^\sigma_{\rho\sigma}$$

Untuk $\mu = 0$

$$\begin{aligned} R_{00} &= \partial_0 \Gamma^\sigma_{0\sigma} - \partial_\sigma \Gamma^\sigma_{00} + \Gamma^\rho_{0\sigma}\Gamma^\sigma_{\rho 0} - \Gamma^\rho_{00}\Gamma^\sigma_{\rho\sigma} \\ &= \left\{ -\nu'' + \nu'\lambda' - \nu'^2 - \frac{2\nu'}{r} \right\} e^{(2\nu - 2\lambda)} \end{aligned} \quad (3.14)$$

Untuk $\mu = 1$

$$\begin{aligned} R_{11} &= \partial_1 \Gamma^\sigma_{1\sigma} - \partial_\sigma \Gamma^\sigma_{11} + \Gamma^\rho_{1\sigma}\Gamma^\sigma_{\rho 1} - \Gamma^\rho_{11}\Gamma^\sigma_{\rho\sigma} \\ &= \nu'' + (\nu')^2 - \lambda'\nu' - \frac{2\lambda'}{r} \end{aligned} \quad (3.15)$$

Untuk $\mu = 2$

$$\begin{aligned} R_{22} &= \partial_2 \Gamma^\sigma_{2\sigma} - \partial_\sigma \Gamma^\sigma_{22} + \Gamma^\rho_{2\sigma}\Gamma^\sigma_{\rho 2} - \Gamma^\rho_{22}\Gamma^\sigma_{\rho\sigma} \\ &= (1 + r\nu' - r\lambda')\, e^{(-2\lambda)} - 1 \end{aligned} \quad (3.16)$$

Untuk $\mu = 3$

$$\begin{aligned} R_{33} &= \partial_3 \Gamma^\sigma_{3\sigma} - \partial_\sigma \Gamma^\sigma_{33} + \Gamma^\rho_{3\sigma}\Gamma^\sigma_{\rho 3} - \Gamma^\rho_{33}\Gamma^\sigma_{\rho\sigma} \\ &= \sin^2\theta \left[ (1 + r\nu' - r\lambda')\, e^{(-2\lambda)} - 1 \right] \\ &= \sin^2\theta\, R_{22} \end{aligned} \quad (3.17)$$

Untuk kondisi dimana tidak ada materi dan energi (vakum), ($R_{\mu\nu} = 0$), maka

$$-\nu'' + \nu'\lambda' - \nu'^2 - \frac{2}{r}\nu' = 0 \tag{3.18}$$

$$\nu'' - \nu'\lambda' + \nu'^2 - \frac{2}{r}\lambda' = 0 \tag{3.19}$$

$$(1 + r\nu' - r\lambda')\, e^{(-2\lambda)} = 1 \tag{3.20}$$

Kemudian pers.(3.18) dan pers.(3.19) dijumlahkan sehingga

$$\frac{-2}{r}\,(\nu' + \lambda') = 0$$

atau

$$\begin{aligned} (\nu' + \lambda') &= 0 \\ \nu + \lambda &= \text{konstan} \end{aligned} \tag{3.21}$$

Pada $r \to \infty$, metrik harus kembali pada bentuk Minkowski $\eta_{\mu\nu}$ sehingga $\nu$ dan $\lambda \to 0$.
maka

$$\nu + \lambda = 0 \to \nu = -\lambda \tag{3.22}$$

dengan memasukkan pers.(3.22) ke pers.(3.20), maka

$$(1 + 2r\nu')\, e^{(2\nu)} = \frac{d}{dr}\left[re^{(2\nu)}\right] = 1 \tag{3.23}$$

Kemudian pers.(3.23) diintegralkan sehingga

$$\int d\left[re^{(2\nu)}\right] = \int dr\; re^{(2\nu)} = r + C \tag{3.24}$$

dengan $C$ merupakan konstanta integrasi yang mempunyai nilai (lihat lampiran A.5)

$$C = -\frac{2GM}{c^2} \equiv -2m \tag{3.25}$$

sehingga metriknya akan menjadi

$$ds^2 \quad = \quad -\left(1-\frac{2m}{r}\right)c^2\left(dt\right)^2+\left(1-\frac{2m}{r}\right)^{-1}\left(dr\right)^2+r^2d\Omega^2 \tag{3.26}$$

metrik pada pers.(3.26) tersebut merupakan metrik Schwarzschild.

## 3.2 Solusi Reissner-Nordstrom

Solusi Schwazschild adalah solusi untuk Persamaan Medan Einstein untuk kasus statik dan simetri bola dari massa $M$. Konsekuensi dari asumsi ini adalah tensor energi momentum $T_{\mu\nu}$ bernilai 0. Jika massa $M$ tersebut bermuatan total $q$, maka nilai tensor energi momentum $T_{\mu\nu}$ tidak sama dengan nol tetapi merupakan tensor energi momentum untuk medan elektromagnetik yang disebabkan oleh muatan total $q$. Dari pers.(2.44) yaitu persamaan medan Einstein

$$R_{\mu\nu} \quad = \quad \frac{8\pi G}{c^4}\left(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T\right) \tag{3.27}$$

dengan tensor energi-momentum yaitu pers.(2.40)

$$T_{\mu\nu}=-F^{\gamma}_{\nu}F_{\mu\gamma}+\frac{1}{4}g_{\mu\nu}F^{\gamma\lambda}F_{\gamma\lambda} \tag{3.28}$$

serta $F_{\mu\nu}$ merupakan tensor kuat medan yaitu

$$F_{\mu\nu}=\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu} \tag{3.29}$$

Nilai $T$ dari pers.(3.27) di atas adalah nol sehingga pers.(3.27) menjadi

$$R_{\mu\nu} \quad = \quad \frac{8\pi G}{c^4}T_{\mu\nu} \tag{3.30}$$

Dari pers.(3.30) di atas, nilai-nilai dari tensor Ricci sama dengan solusi Schwarzschild pada subbab 3.1 di atas. Untuk nilai-nilai dari tensor energi momentum adalah (lihat lampiran A.6)

$$\begin{aligned} T_{00} &= -\frac{1}{2}f^2(e^{-2\lambda}) \\ T_{11} &= \frac{1}{2}f^2e^{-2\nu} \\ T_{22} &= -\frac{1}{2}r^2f^2e^{-2\lambda-2\nu} \\ T_{33} &= -\frac{1}{2}r^2f^2e^{-2\lambda-2\nu}\sin^2\theta \end{aligned} \tag{3.31}$$

Dengan menggunakan cara yang sama pada solusi Schwarzschild, akan didapatkan metrik Reissner-Nordstrom sebagai berikut:

$$ds^2 = -\left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right)du^2+\left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right)^{-1}dr^2+r^2d\Omega^2 \tag{3.32}$$

dengan

$$Q^2 \equiv -\frac{q^2G}{8\pi\epsilon_0^2c^4} \tag{3.33}$$

Metrik di atas merupakan generalisasi dari metrik Schwarzschild. Jika diambil $Q=0$ pada metrik di atas yang berarti massa $M$ menjadi tidak bermuatan, maka metrik di aas akan kembali pada bentuk metrik Schwarzschild.

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# BAB 4

# SOLUSI BEROTASI PERSAMAAN MEDAN EINSTEIN

Solusi Schwarzschild dan solusi Reissner Nordstrom adalah solusi untuk persamaan medan Einstein yang statik dan simetri bola yang masing-masing bersesuaian dengan massa $M$ yang tidak bermuatan dan massa $M$ yang bermuatan $q$. Karena pada umumnya benda langit berotasi, maka diperlukan adanya solusi untuk massa $M$ bermuatan $q$ yang berotasi guna men-generalisasi solusi Schwarzschild dan solusi Reissner Nordstrom. Pada subbab berikut akan dibahas bagaimana solusi persamaan medan Einstein untuk kasus berotasi dengan menggunakan Algoritma Newman-Janis. Jika algoritma Newman-Janis diterapkan pada metrik Scwarzschild, maka akan didapatkan solusi Kerr sedangkan jika algoritma Newman-Janis diterapkan pada metrik Reissner-Nordstrom, maka akan didapatkan solusi Kerr-Newman.

## 4.1 Solusi Kerr

Untuk memperoleh solusi Kerr dengan Algoritma Newman-Janis, maka langkah yang dilakukan yaitu membentuk metrik Schwarzschild dalam koordinat null yaitu dengan menerapkan transformasi $u = t - r^*$ dengan $r^*$ adalah koordinat *tortoise* yang diberikan oleh

$$dr = \left(1 - \frac{2m}{r}\right) dr^* \tag{4.1}$$

sehingga bentuk metrik Schwarzschild dalam ungkapan koordinat null menjadi:

$$ds^2 = -\left(1 - \frac{2m}{r}\right) du^2 - 2du\, dr + r^2 d\Omega^2 \tag{4.2}$$

Tensor metrik dari elemen garis di atas adalah:

$$g_{\mu\nu} = \begin{pmatrix} -\left(1-\frac{2m}{r}\right) & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{4.3}$$

dengan bentuk kontravariannya adalah:

$$g_{\mu\nu} \quad = \quad \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & \left(1-\frac{2m}{r}\right) & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix} \tag{4.4}$$

Komponen-komponen vektor-4 null yang bersesuaian dengan tensor metrik di atas adalah:

$$\begin{aligned} l^\mu &= (0,-1,0,0) \\ n^\mu &= \left(-1, \frac{1}{2}\left(1-\frac{2m}{r}\right), 0, 0\right) \\ m^\mu &= \frac{1}{r\sqrt{2}}\left(0,0,1,\frac{i}{\sin\theta}\right) \\ \bar{m}^\mu &= \frac{1}{r\sqrt{2}}\left(0,0,1,\frac{i}{\sin\theta}\right) \end{aligned} \tag{4.5}$$

atau

$$\begin{aligned} l^\mu\partial_\mu &= -\partial_r \\ n^\mu\partial_\mu &= -\partial_u + \frac{1}{2}\left(1-\frac{2m}{r}\right)\partial_r \\ m^\mu\partial_\mu &= \frac{1}{r\sqrt{2}}\left(\partial_\theta + \frac{i}{\sin\theta}\partial_\phi\right) \\ m^\mu\partial_\mu &= \frac{1}{r\sqrt{2}}\left(\partial_\theta - \frac{i}{\sin\theta}\partial_\phi\right) \end{aligned} \tag{4.6}$$

Kemudian nilai $r$ ditransformasikan menjadi bentuk kompleks yaitu $r \to \tilde{r}$. Dengan menghilangkan tanda $\tilde{}$, maka bentuk vektor-4 null pers.(4.5) di atas menjadi

$$\begin{aligned} l^\mu &= (0,-1,0,0) \\ n^\mu &= \left(-1, \frac{1}{2}\left(1-\frac{m}{r}-\frac{m}{\bar{r}}\right), 0, 0\right) \\ m^\mu &= \frac{1}{\bar{r}\sqrt{2}}\left(0,0,1,\frac{i}{\sin\theta}\right) \\ \bar{m}^\mu &= \frac{1}{r\sqrt{2}}\left(0,0,1,\frac{i}{\sin\theta}\right) \end{aligned} \tag{4.7}$$

atau

$$\begin{aligned} l^\mu \partial_\mu &= -\partial_r \\ n^\mu \partial_\mu &= -\partial_u + \frac{1}{2}\left(1-\frac{m}{r}-\frac{m}{\bar{r}}\right)\partial_r \\ m^\mu \partial_\mu &= \frac{1}{\bar{r}\sqrt{2}}\left(\partial_\theta + \frac{i}{\sin\theta}\partial_\phi\right) \\ \bar{m}^\mu \partial_\mu &= \frac{1}{r\sqrt{2}}\left(\partial_\theta - \frac{i}{\sin\theta}\partial_\phi\right) \end{aligned} \tag{4.8}$$

Kemudian dilakukan transformasi

$$x^\rho \to x'^\rho = x^\rho + ia\cos\theta\,(\delta_0^\rho - \delta_1^\rho) \tag{4.9}$$

sehingga diperoleh

$$\begin{aligned} u \to u' &= u + ia\cos\theta \\ r \to r' &= r - ia\cos\theta \\ \theta \to \theta' &= \theta \\ \phi \to \phi' &= \phi \end{aligned} \tag{4.10}$$

atau

$$\begin{aligned} u &= u' - ia\cos\theta = u' - ia\cos\theta' \\ r &= r' + ia\cos\theta = r' + ia\cos\theta' \end{aligned} \tag{4.11}$$

dengan $v', r', \theta', \phi'$ adalah kuantitas riil. Vektor-vektor basis bertransformasi dengan menggunakan aturan rantai:

$$\frac{\partial}{\partial x^{\mu}} \to \frac{\partial}{\partial x'^{\mu}} = \frac{\partial x^{\nu}}{\partial x'^{\mu}}\frac{\partial}{\partial x^{\nu}}$$

atau dapat dituliskan :

$$\partial_{\mu} \to \partial_{\mu'} = \frac{\partial x^{\nu}}{\partial x'^{\mu}}\partial_{\nu} \tag{4.12}$$

Setelah ditransformasi dengan menghilangkan tanda ', hasil transformasinya adalah:

$$\begin{aligned} l^{\mu}\partial_{\mu} &= -\partial_r \\ n^{\mu}\partial_{\mu} &= -\partial_u + \frac{1}{2}\left(1 - \frac{2mr}{r^2 + a^2\cos^2\theta}\right)\partial_r \\ m^{\mu}\partial_{\mu} &= \frac{1}{\sqrt{2}(r - ia\cos\theta)}\left(ia\sin\theta\partial_v - ia\sin\theta\partial_r + \partial_\theta + \frac{i}{\sin\theta}\partial_\phi\right) \\ \bar{m}^{\mu}\partial_{\mu} &= \frac{1}{\sqrt{2}(r + ia\cos\theta)}\left(-ia\sin\theta\partial_v + ia\sin\theta\partial_r + \partial_\theta - \frac{i}{\sin\theta}\partial_\phi\right) \end{aligned} \tag{4.13}$$

atau

$$
\begin{aligned}
l^{\mu} &= (0,-1,0,0) \\
n^{\mu} &= \left(-1, \frac{1}{2}\left(1-\frac{2mr}{r^2+a^2\cos^2\theta}\right), 0, 0\right) \\
m^{\mu} &= \frac{1}{\sqrt{2}(r-ia\cos\theta)}\left(ia\sin\theta, -ia\sin\theta, 1, \frac{i}{\sin\theta}\right) \\
\bar{m}^{\mu} &= \frac{1}{\sqrt{2}(r+ia\, cos\theta)}\left(-ia\sin\theta, ia\sin\theta, 1, -\frac{i}{\sin\theta}\right)
\end{aligned}
\tag{4.14}
$$

Selanjutnya komponen-komponen tensor metrik kontravarian $g^{\mu\nu}$ dapat dibentuk yaitu:

$$
g^{\mu\nu} = -l^{\mu}n^{\nu} - l^{\mu}n^{\nu} + m^{\mu}\bar{m}^{\nu} + m^{\nu}\bar{m}^{\mu} \tag{4.15}
$$

Maka tensor metrik Kontravariannya adalah:

$$
g^{\mu\nu} = \begin{pmatrix}
-\left(1-\frac{2mr}{\rho^2}\right) & -1 & 0 & -\frac{2mr\ a\sin^2\theta}{\rho^2} \\
-1 & 0 & 0 & a\sin^2\theta \\
 & 0 & \rho^2 & 0 \\
-\frac{2mr\ a\sin^2\theta}{\rho^2} & a\sin^2\theta & 0 & -\frac{sin^2\theta}{\rho^2}\left(a^2\sin^2\theta\Delta-\left(r^2+a^2\right)^2\right)
\end{pmatrix}
$$

sehingga elemen garisnya menjadi:

$$
\begin{aligned}
ds^2 &= -\left(1-\frac{2mr}{\rho^2}\right)du^2 - 2du\, dr - 4mr\ a\ \frac{\sin^2\theta}{\rho^2}du\, d\phi \\
&\quad +2a\sin^2\theta dr\, d\phi + \rho^2 d\theta^2 - \frac{sin^2\theta}{\rho^2}\left(a^2\sin^2\theta\Delta - \left(r^2+a^2\right)d\phi^2\right.
\end{aligned}
\tag{4.16}
$$

dengan

$$\rho^2 = r^2 + a^2 \cos^2\theta \tag{4.17}$$
$$\Delta = r^2 + a^2 - 2mr \tag{4.18}$$

Pers.(4.16) merupakan metrik kerr yang menggambarkan massa $M$ yang berotasi. Parameter rotasi ditunjukkan dengan adanya suku $a$ yang sebanding dengan momentum sudut.

## 4.2 Solusi Kerr-Newman

Solusi Reissner-Nordstrom adalah solusi untuk lubang hitam yang statik, simetri bola dan bermuatan. Sama seperti analogi dari generalisasi solusi Schwarzschild ke solusi Kerr, solusi Reissner-Nordstrom juga dapat digeneralisasi ke solusi Kerr-Newman menggunakan Algoritma Newman-Janis. Metrik Reissner Nordstrom ditransformasi dengan menggunakan $u = t - r^*$, dengan $r^*$ adalah tortoise koordinat yang memenuhi

$$dr = \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right) dr^* \tag{4.19}$$

sehingga bentuk metrik Reissner-Nordstrom menjadi:

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right) du^2 - 2du\, dr + r^2 d\Omega^2 \tag{4.20}$$

Tensor metrik dari elemen garis di atas adalah:

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right) & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \tag{4.21}$$

dengan bentuk kontravariannya adalah:

$$g_{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right) & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2 \sin^2\theta} \end{pmatrix} \tag{4.22}$$

Komponen-komponen vektor-4 null yang bersesuaian dengan tensor metrik di atas adalah:

$$\begin{aligned} l^\mu &= (0, -1, 0, 0) \\ n^\mu &= \left(-1, \frac{1}{2}\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right), 0, 0\right) \\ m^\mu &= \frac{1}{r\sqrt{2}}\left(0, 0, 1, \frac{i}{sin\theta}\right) \\ \bar{m}^\mu &= \frac{1}{r\sqrt{2}}\left(0, 0, 1, \frac{i}{\sin\theta}\right) \end{aligned} \tag{4.23}$$

atau

$$\begin{aligned} l^\mu \partial_\mu &= -\partial_r \\ n^\mu \partial_\mu &= -\partial_u + \frac{1}{2}\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)\partial_r \\ m^\mu \partial_\mu &= \frac{1}{r\sqrt{2}}\left(\partial_\theta + \frac{i}{\sin\theta}\partial_\phi\right) \\ m^\mu \partial_\mu &= \frac{1}{r\sqrt{2}}\left(\partial_\theta - \frac{i}{\sin\theta}\partial_\phi\right) \end{aligned} \tag{4.24}$$

Kemudian nilai $r$ ditransformasikan menjadi bentuk kompleks yaitu $r \rightarrow \tilde{r} = r + iy$. Dengan menghilangkan tanda aksen, maka bentuk vektor-4 null di atas adalah:

$$
\begin{aligned}
l^\mu &= (0,-1,0,0) \\
n^\mu &= \left(-1, \frac{1}{2}\left(1-\frac{m}{r}-\frac{m}{\bar{r}}+\frac{Q^2}{r\bar{r}}\right),0,0\right) \\
m^\mu &= \frac{1}{\bar{r}\sqrt{2}}\left(0,0,1,\frac{i}{\sin\theta}\right) \\
\bar{m}^\mu &= \frac{1}{r\sqrt{2}}\left(0,0,1,\frac{i}{\sin\theta}\right)
\end{aligned}
\tag{4.25}
$$

atau

$$
\begin{aligned}
l^\mu \partial_\mu &= -\partial_r \\
n^\mu \partial_\mu &= -\partial_u + \frac{1}{2}\left(1-\frac{m}{r}-\frac{m}{\bar{r}}+\frac{Q^2}{r\bar{r}}\right)\partial_r \\
m^\mu \partial_\mu &= \frac{1}{\bar{r}\sqrt{2}}\left(\partial_\theta + \frac{i}{\sin\theta}\partial_\phi\right) \\
\bar{m}^\mu \partial_\mu &= \frac{1}{r\sqrt{2}}\left(\partial_\theta - \frac{i}{\sin\theta}\partial_\phi\right)
\end{aligned}
\tag{4.26}
$$

setelah itu dilakukan transformasi

$$
x^\rho \to x'^\rho = x^\rho + ia\cos\theta\left(\delta_0^\rho - \delta_1^\rho\right) \tag{4.27}
$$

sehingga diperoleh

$$
\begin{aligned}
u \to u' &= u + ia\cos\theta \\
r \to r' &= r - ia\cos\theta \\
\theta \to \theta' &= \theta \\
\phi \to \phi' &= \phi
\end{aligned}
\tag{4.28}
$$

atau

$$
\begin{aligned}
u &= u' - ia\cos\theta = u' - ia\cos\theta' \\
r &= r' + ia\cos\theta = r' + ia\cos\theta'
\end{aligned}
\tag{4.29}
$$

dengan $v', r', \theta', \phi'$ adalah kuantitas riil. Vektor-vektor basis bertransformasi dengan menggunakan aturan rantai:

$$
\frac{\partial}{\partial x^{\mu}} \rightarrow \frac{\partial}{\partial x'^{\mu}} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \frac{\partial}{\partial x^{\nu}}
$$

atau dapat dituliskan :

$$
\partial_{\mu} \rightarrow \partial_{\mu'} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \partial_{\nu} \tag{4.30}
$$

Setelah ditransformasi dengan menghilangkan tanda ', hasil transformasinya adalah:

$$
\begin{aligned}
l^{\mu}\partial_{\mu} &= -\partial_r \\
n^{\mu}\partial_{\mu} &= -\partial_u + \frac{1}{2}\left(1 - \frac{2mr - Q^2}{r^2 + a^2 cos^2\theta}\right)\partial_r \\
m^{\mu}\partial_{\mu} &= \frac{1}{\sqrt{2}(r - ia\ \cos\theta)}\left(ia\sin\theta\partial_v - ia\sin\theta\partial_r + \partial_{\theta} + \frac{i}{\sin\theta}\partial_{\phi}\right) \\
\bar{m}^{\mu}\partial_{\mu} &= \frac{1}{\sqrt{2}(r + ia\cos\theta)}\left(-ia\sin\theta\partial_v + ia\sin\theta\partial_r + \partial_{\theta} - \frac{i}{\sin\theta}\partial_{\phi}\right)
\end{aligned}
\tag{4.31}
$$

atau

$$\begin{aligned}
l^{\mu} &= (0,-1,0,0) \\
n^{\mu} &= \left(-1, \frac{1}{2}\left(1-\frac{2mr-Q^2}{r^2+a^2\cos^2\theta}\right), 0, 0\right) \\
m^{\mu} &= \frac{1}{\sqrt{2}(r-ia\cos\theta)}\left(ia\sin\theta, -ia\sin\theta, 1, \frac{i}{\sin\theta}\right) \\
\bar{m}^{\mu} &= \frac{1}{\sqrt{2}(r+ia\cos\theta)}\left(-ia\sin\theta, ia\sin\theta, 1, -\frac{i}{\sin\theta}\right)
\end{aligned} \tag{4.32}$$

Selanjutnya komponen-komponen tensor metrik kontravarian $g^{\mu\nu}$ dapat dibentuk yaitu:

$$g^{\mu\nu} = -l^{\mu}n^{\nu} - l^{\mu}n^{\nu} + m^{\mu}\bar{m}^{\nu} + m^{\nu}\bar{m}^{\mu} \tag{4.33}$$

dengan tensor metrik Kontravariannya adalah:

$$g^{\mu\nu} = \begin{pmatrix} -\left(1-\frac{2mr-Q^2}{\rho^2}\right) & -1 & 0 & -\frac{(2mr-Q^2)a\sin^2\theta}{\rho^2} \\ -1 & 0 & 0 & a\sin^2\theta \\ & 0 & \rho^2 & 0 \\ -\frac{(2mr-Q^2)a\sin^2\theta}{\rho^2} & a\sin^2\theta & 0 & -\frac{sin^2\theta}{\rho^2}\left(a^2sin^2\theta\Delta-\left(r^2+a^2\right)^2\right) \end{pmatrix}$$

sehingga elemen garisnya menjadi:

$$\begin{aligned}
ds^2 &= -\left(1-\frac{2mr-Q^2}{\rho^2}\right)du^2 - 2du\,dr - 2a\sin^2\frac{(2mr-Q^2)\theta}{\rho^2}du\,d\phi \\
&\quad +2a\sin^2\theta dr\,d\phi + \rho^2 d\theta^2 - \frac{\sin^2\theta}{\rho^2}\left(a^2\sin^2\theta\Delta - \left(r^2+a^2\right)d\phi^2\right.
\end{aligned} \tag{4.34}$$

Bentuk metrik di atas disebut metrik Kerr-Newman.

# BAB 5

# DISKUSI

Persamaan Medan Einstein memiliki banyak solusi tetapi dalam tugas akhir ini hanya dirumuskan ulang solusi untuk 4 macam lubang hitam. Jika solusinya adalah solusi statik (tidak ada momentum angular), maka solusinya adalah metrik Schwarzschild dan Reissner-Nordstrom yang masing-masing mencirikan lubang hitam bermassa $m$ tanpa muatan listrik, dan lubang hitam bermassa $m$ dengan muatan $q$. Jika solusinya adalah solusi yang berotasi, maka solusinya adalah metrik Kerr dan Kerr-Newman yang masing-masing mencirikan lubang hitam bermassa $m$ dengan momentum angular $J$ tanpa muatan listrik, dan lubang hitam bermassa $m$ dengan dengan momentum angular $J$ dan bermuatan $q$. Keempat metrik ini adalah solusi lubang hitam dari relativitas umum. Hubungan 4 metrik ini adalah seperti pada tabel berikut

**Tabel 5.1:** Kedudukan lubang hitam bermuatan dan berotasi

| | Tidak berotasi (J=0) | Berotasi (J$\neq$ 0) |
|---|---|---|
| Tidak bermuatan (Q=0) | **Schwarzschild** | **Kerr** |
| Bermuatan (Q$\neq$ 0) | **Reissner-Nordstrom** | **Kerr-Newman** |

Bentuk metrik dari koordinat null pers.(4.34) dapat ditransformasi ke bentuk *Boyer-Lindquist* dengan menerapkan transformasi berikut:

$$dt = du + \frac{r^2 + a^2}{\Delta} dr, \quad d\psi = d\phi + \frac{a}{\Delta} dr \tag{5.1}$$

maka diperoleh metrik Kerr-Newman dalam koordinat Boyer-Lindquist:

$$\begin{aligned} ds^2 &= -\frac{\Delta}{\rho^2}\left(a\sin^2\theta d\psi - dt\right)^2 + \frac{\sin^2\theta}{\rho^2}\left((r^2+a^2)d\psi - a\, dt\right)^2 \\ &\quad + \frac{\rho^2}{\Delta^2} dr^2 + \rho^2 d\theta^2 \end{aligned} \tag{5.2}$$

Dalam bab diskusi ini akan dikaji mengenai sifat-sifat di sekitar ruang waktu Kerr-Newman. Sifat yang akan dikaji antara lain tentang horison peristiwa Event Horizon (EH) dan pergeseran merah permukaan (Surface of Infinite Red-shift (SIR)). EH dapat dicari dari permukaan *hyper* null. Persamaan permukaan *hyper* null adalah

$$f(x^0, x^1, x^2, x^3) = 0 \tag{5.3}$$

dengan vektor normal yang merupakan vektor kovarian

$$n_\mu = \frac{\partial f}{\partial x^\mu} \tag{5.4}$$

Koordinat Boyer-Lindquist mempunyai singularitas di $\Delta = 0$ sehingga

$$\begin{aligned} r^2 - 2mr + a^2 + Q^2 &= 0 \\ r = r_\pm &= m \pm \sqrt{m^2 - a^2 - Q^2} \end{aligned} \tag{5.5}$$

dengan $r_+$ merupakan solusi untuk daerah di luar sumber massa dan $r_-$ untuk daerah di dalam sumber massa. Pada diskusi ini hanya dibahas untuk $r_+$ saja (diluar sumber massa)

$$r = r_+ = m + \sqrt{m^2 - a^2 - Q^2} \tag{5.6}$$

sehingga persamaan permukaan *hyper* adalah

$$\begin{aligned} f &= r - r_+ \\ &= r - m - \sqrt{m^2 - a^2 - Q^2} \end{aligned} \tag{5.7}$$

dengan nilai $n_\mu$ adalah

$$n_\mu = \frac{\partial f}{\partial x^\mu} = (0, 1, 0, 0) \tag{5.8}$$

Permukaan *hyper* akan null jika $n^{\mu}n_{\mu} = 0$, tetapi

$$\begin{aligned} n^{\mu}n_{\mu} &= g^{\mu\nu}n_{\nu}n_{\mu} \\ &= g^{11}n_1n_1 \\ &= g^{11} \\ &= \frac{\Delta}{\rho^2} \end{aligned} \tag{5.9}$$

sehingga $n^{\mu}n_{\mu}$ akan sama dengan nol jika $\Delta = 0$ atau $r = r_+$. Karena $n^{\mu}n_{\mu} = 0$ pada $r = r_+$, maka $r = r_+$ adalah permukaan *hyper* null. Cahaya yang merambat melewati batas ini setelah melalui permukaan ini tidak akan pernah keluar melalui permukaan ini. permukaan ini disebut sebbagai event horizon sehingga event horizon untuk solusi Kerr-Newman adalah :

$$r = r_+ = m + \sqrt{m^2 - a^2 - Q^2} \tag{5.10}$$

Cahaya yang bergerak menjauh di sekitar permukaan *hyper* null akan mengalami pergeseran merah gravitasional, artinya panjang gelombang cahaya akan semakin memanjang menjadi lebih besar. Besar fraksi perubahan panjang gelombang ini dinyatakan dengan konstanta $z$ yang tidak berdimensi yaitu

$$z = \frac{\lambda_0 - \lambda_e}{\lambda_e} = \frac{\lambda_0}{\lambda_e} - 1 \tag{5.11}$$

dengan $c = \lambda v$, maka

$$\begin{aligned} z &= \frac{\frac{c}{v_0}}{\frac{c}{v_e}} - 1 \\ &= \frac{v_e}{v_0} - 1 \end{aligned} \tag{5.12}$$

dengan $\nu_e$ adalah besarnya frekuensi yang dipancarkan dan $\nu_o$ adalah frekuensi yang diterima observer (pengamat) di luar event horizon. Misalkan cahaya dipancarkan di $r_2$ dan diterima pengamat di $r_1$, maka

$$z = \frac{v_2}{v_1} - 1 = \sqrt{\frac{g_{00}(r_1)}{g_{00}(r_2)}} - 1 \tag{5.13}$$

dengan nilai $g_{00}(r)$ untuk ruang-waktu Kerr-Newman adalah

$$g_{00}(r) = -\left(1 - \frac{2mr - Q^2}{\rho^2}\right) \tag{5.14}$$

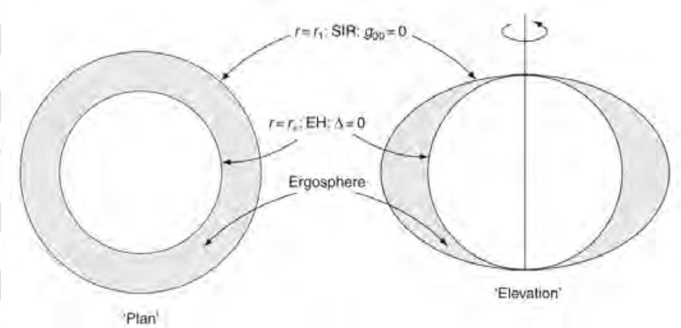
$r_1$ diasumsikan sangat jauh dari sumber $(r \to \infty)$ sehingga tensor metrik akan kembali ke ruang-waktu datar (Minkowskian):

$$g_{00}(r_1) = -1 \tag{5.15}$$

sehingga

$$\begin{aligned} z &= \frac{1}{\sqrt{-g_{00}(r_2)}} - 1 \\ &= \sqrt{\frac{1}{1 - \frac{2mr - Q^2}{\rho^2}}} - 1 \\ &= \sqrt{\frac{\rho^2}{\rho^2 - 2mr - Q^2}} - 1 \end{aligned} \tag{5.16}$$

Saat $\rho^2 \to 2mr + Q^2$, maka $z \to \infty$ dan nilai $g_{00}$ pada pers.(5.13) menjadi




**Gambar 5.1:** Ergosphere dan SIR pada lubang hitam Kerr-Newman

$g_{00} = 0$ sehingga pers.(5.14) menjadi

$$
\begin{aligned}
\left(1 - \frac{2mr - Q^2}{\rho^2}\right) &= 0 \\
\frac{2mr - Q^2}{\rho^2} &= 1 \\
2mr - Q^2 &= \rho^2 \\
0 &= r^2 + a^2 cos^2\theta - 2mr + Q^2 \\
r_{1,2} &= m \pm \sqrt{m^2 - a^2 cos^2\theta - Q^2} \qquad (5.17)
\end{aligned}
$$

Di sini hanya diambil solusi untuk $r = r_1$ saja. Posisi $r_1$ berada di luar event horizon sedangkan posisi $r_2$ berada di dalam event horizon. Permukaan pada $r = r_1$ disebut sebagai *Surface Infinite Redshift (SIR)* seperti ditunjukan pada Gambar 5.1 diatas. Ergosphere adalah daerah diantara keduanya yang didefinisikan oleh :

$$m + \sqrt{m^2 - a^2 - Q^2} = r_+ < r < r_1 = m + \sqrt{m^2 - a^2 cos^2\theta - Q^2} \qquad (5.18)$$

Sekarang misalkan terdapat partikel yang diam pada Ergosphere. Partikel yang dimaksud dalam skala alam semesta adalah bintang atau galaksi. Partikel yang diam pada ergosphere berarti komponen $r, \theta, \phi$ bernilai konstan atau $dr = d\theta = d\phi = 0$ dan partikel hanya akan bergerak mengikuti lintasan geodesik timelikenya, yaitu

$$ds^2 = -\left(1 - \frac{2mr - Q^2}{\rho^2}\right) c^2 dt^2 \tag{5.19}$$

di dalam Ergosphere dengan $r < r_1$, maka

$$\begin{aligned} r &< r_1 \\ r &< m + \sqrt{m^2 - a^2 cos^2\theta - Q^2} \\ r - m &< \sqrt{m^2 - a^2 cos^2\theta - Q^2} \\ r^2 - 2mr + m^2 &< m^2 - a^2 cos^2\theta - Q^2 \\ r^2 + a^2 cos^2\theta + Q^2 &< 2mr \end{aligned} \tag{5.20}$$

atau

$$\begin{aligned} \rho^2 + Q^2 &< 2mr \\ \rho^2 &< 2mr - Q^2 \end{aligned} \tag{5.21}$$

sehingga

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2mr - Q^2}{\rho^2}\right) c^2 dt^2 \\ ds^2 &> 0 \end{aligned} \tag{5.22}$$

Ada yang kurang dari hasil ini karena untuk partikel real $ds^2 < 0$, maka diasumsikan terdapat setidaknya 1 koordinat yang tidak nol antara $r, \theta$ dan $\phi$. Diasumsikan partikel berada di $r$ dan $\theta$ yang tertentu, tetapi bergerak kearah $\phi$ positif, maka kecepatan 4-partikel adalah

$$\begin{aligned} u^\mu = \frac{dx^\mu}{d\tau} &= (u^0, u^1, u^2, u^3) \\ &= (u^0, 0, 0, u^3) \\ &= u^0 \left(1, 0, 0, \frac{u^3}{u^0}\right) \\ &= u^0 \left(1, 0, 0, \frac{\Omega}{c}\right) \end{aligned} \tag{5.23}$$

dengan $\Omega = \frac{d\phi}{dt} = \frac{d\phi}{d\tau}\frac{d\tau}{dt} = c\frac{u^3}{u^0}$ kondisi untuk $u^\mu$ timelike adalah

$$ds^2 = g_{\mu\nu}u^\mu u^\nu < 0 \tag{5.24}$$

maka

$$g_{00}(u^0)^2 + 2g_{03}u_0u^3 + g_{33}(u^3)^2 < 0 \tag{5.25}$$

atau

$$\begin{aligned} \frac{g_{00}(u^0)^2}{g_{33}(u^0)^2} + 2g_{03}\frac{u_0u^3}{g_{33}(u^0)^2} + \frac{g_{33}(u^3)^2}{g_{33}(u^0)^2} &< 0 \\ \frac{g_{00}}{g_{33}} + 2\frac{g_{03}}{g_{33}}\frac{\Omega}{c} + \frac{\Omega^2}{c^2} &< 0 \end{aligned} \tag{5.26}$$

solusi untuk $\frac{g_{00}}{g_{33}} + 2\frac{g_{03}}{g_{33}}\frac{\Omega}{c} + \frac{\Omega^2}{c^2} = 0$ adalah

$$\begin{aligned} \frac{\Omega}{c} &= \frac{-2\frac{g_{00}}{g_{33}} \pm \sqrt{\left(2\frac{g_{00}}{g_{33}}\right)^2 - 4.1.\frac{g_{00}}{g_{33}}}}{2.1} \\ &= -\frac{g_{03}}{g_{33}} \pm \sqrt{\left(\frac{g_{03}}{g_{33}}\right)^2 - \frac{g_{00}}{g_{33}}} \end{aligned} \tag{5.27}$$

Misalkan diambil

$$-\frac{g_{03}}{g_{33}} = \frac{\omega}{c} \tag{5.28}$$

sehingga terdapat 2 solusi yaitu

$$\begin{aligned} \Omega_{min} &= \omega - \sqrt{\omega^2 - c^2\frac{g_{00}}{g_{33}}} \\ \Omega_{max} &= \omega + \sqrt{\omega^2 - c^2\frac{g_{00}}{g_{33}}} \end{aligned} \tag{5.29}$$

maka agar pertidaksamaan (5.26) terpenuhi, nilai $\Omega$ harus berada pada rentang

$$\Omega_{min} < \Omega < \Omega_{max} \tag{5.30}$$

dengan $\Omega$ adalah kecepatan angular partikel pada Ergosphere Kerr-Newman yang nilainya bergantung pada $r$ karena tensor metriknya bergantung pada $r$. Karena $g_{00} < 0$, maka $\Omega_{min} < 0$ sehingga partikel bisa bergerak berputar melawan arah rotasi lubang hitam. Ketika $r$ mendekati SIR, maka $g_{00} \to 0$ dan $\Omega_{min} = 0$ dan $\Omega_{max} = 2\omega$ sehingga partikel tidak bisa lagi untuk bergerak melawan arah rotasi dari lubang hitam, tetapi partikel masih bisa diam (menurut pengamat yang jauh dari massa sumber). Kemudian jika partikel telah masuk ke dalam ergosfer, maka $\Omega = 0$ sehingga partikel tidak bisa lagi mempertahankan keadaan diamnya dan akan bergerak rotasi searah dengan arah rotasi sumber massa. Dengan kata lain, SIR merupakan batas partikel untuk masih bisa mempertahankan kedudukan diamnya.

# BAB 6

# PENUTUP

## 6.1 Kesimpulan

Dalam tugas akhir ini, telah diturunkan solusi lubang hitam untuk persamaan medan Einstein yaitu untuk solusi statik adalah solusi Schwarzschild dan solusi Reissner-Nordstrom. Kemudian setelah diterapkan metode Newman-Janis pada kedua metrik tersebut, dihasilkan solusi Reissner-Nordstrom dan solusi Kerr-Newman. Disekitar lubang hitam Kerr-Newman terdapat ergosphere yang membedakannya dengan lubang hitam Schwarzschild dan Reissner-Nordstrom. Partikel berada di dekat horizon dapat bergerak berlawanan arah rotasi lubang hitam, sedangkan partikel yang berada tepat pada SIR partikel tersebut tidak bisa lagi bergerak melawan arah rotasi lubang hitam, dan setelah masuk pada ergosfer, partikel akan bergerak melingkar searah gerakan sumber massa. Dengan kata lain, SIR adalah batas dari partikel untuk dapat mempertahankan keadaan diamnya. Partikel yang dimaksud dalam skala ini adalah bintang atau galaksi. Partikel yang bergerak ke arah massa $M$ dan melewati event horizon tidak bisa lagi bergerak ke arah luar sekalipun itu cahaya. Selanjutnya massa $M$ ini disebut sebagai *blackhole*.

## 6.2 Saran

Saran untuk penelitian selanjutnya yaitu dengan mengkaji sifat-sifat ruang-waktu Kerr-Newman pada daerah di bagian dalam event horizon serta dengan menambahkan partikel uji yang bermuatan.

Jika ditambahkan 1 dimensi ekstra pada ruang-waktu Kerr-Newman, maka salah satu solusi yang didapatkan adanya *Black String* yang bermuatan dan berotasi. Untuk kasus Black String yang tidak bermuatan telah dikerjakan pada arXiv:1205.1656 [gr-qc].

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# LAMPIRAN A

## A.1 Persamaan Geodesik

Lagrangian didefinisikan sebagai fungsi dari koordinat $x^\mu$ dan turunan pertamanya:

$$\mathcal{L} = \mathcal{L}\left(x^\mu, \frac{dx^\mu}{ds}, s\right) \tag{A.1}$$

sehingga integral aksi adalah

$$I = \int_{s_A}^{s_B} \mathcal{L}\left(x^\mu, \frac{dx^\mu}{ds}, s\right) ds \tag{A.2}$$

dengan variasi integral aksi

$$\delta I = \int_{s_A}^{s_B} \delta\mathcal{L}\left(x^\mu, \frac{dx^\mu}{ds}, s\right) ds = 0 \tag{A.3}$$

serta

$$\delta\mathcal{L} = \mathcal{L}\left(x^\mu + \delta x^\mu, \frac{dx^\mu}{ds} + \delta\frac{dx^\mu}{ds}, s\right) - \mathcal{L}\left(x^\mu, \frac{dx^\mu}{ds}, s\right) \tag{A.4}$$

Suku pertama pada pers.(A.4) diekspansikan dalam deret Taylor dengan mengambil suku ke-0 dan ke-1

$$\begin{aligned}
\delta\mathcal{L} &= \mathcal{L}\left(x^\mu, \frac{dx^\mu}{ds}, s\right) + \frac{\partial\mathcal{L}}{\partial x^\mu}\delta x^\mu \\
&\quad + \frac{\partial\mathcal{L}}{\partial\left(\frac{dx^\mu}{ds}\right)}\delta\left(\frac{dx^\mu}{ds}\right) - \mathcal{L}\left(x^\mu, \frac{dx^\mu}{ds}, s\right) \\
&= \frac{\partial\mathcal{L}}{\partial x^\mu}\delta x^\mu + \frac{\partial\mathcal{L}}{\partial\left(\frac{dx^\mu}{ds}\right)}\delta\left(\frac{dx^\mu}{ds}\right) \\
&= \frac{\partial\mathcal{L}}{\partial x^\mu}\delta x^\mu + \frac{\partial\mathcal{L}}{\partial\left(\frac{dx^\mu}{ds}\right)}\frac{d}{ds}\delta x^\mu \\
&= \frac{\partial\mathcal{L}}{\partial x^\mu}\delta x^\mu + \frac{d}{ds}\left[\frac{\partial\mathcal{L}}{\partial\left(\frac{dx^\mu}{ds}\right)}\delta x^\mu\right] \\
&\quad - \frac{d}{ds}\left[\frac{\partial\mathcal{L}}{\partial\left(\frac{dx^\mu}{ds}\right)}\right]\delta x^\mu
\end{aligned} \tag{A.5}$$

pers.(A.5) disubstitusikan ke pers.(A.3)

$$\begin{aligned}
\delta I &= \int_{s_A}^{s_B}\left[\frac{\partial\mathcal{L}}{\partial x^\mu} - \frac{d}{ds}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{dx^\mu}{ds}\right)}\right)\right]\delta x^\mu ds \\
&\quad + \int_{s_A}^{s_B} d\left[\frac{\partial\mathcal{L}}{\partial\left(\frac{dx^\mu}{ds}\right)}\delta x^\mu\right] \\
&= \int_{s_A}^{s_B}\left[\frac{\partial\mathcal{L}}{\partial x^\mu} - \frac{d}{ds}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{dx^\mu}{ds}\right)}\right)\right]\delta x^\mu ds \\
&\quad + \left[\frac{\partial\mathcal{L}}{\partial\left(\frac{dx^\mu}{ds}\right)}\delta x^\mu\right]_{s_A}^{s_B} \\
&= \int_{s_A}^{s_B}\left[\frac{\partial\mathcal{L}}{\partial x^\mu} - \frac{d}{ds}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{dx^\mu}{ds}\right)}\right)\right]\delta x^\mu ds
\end{aligned} \tag{A.6}$$

suku ke-3 pers.(A.6) lenyap karena

$$\delta x^\mu(s_A) = \delta x^\mu(s_B) = 0 \tag{A.7}$$

karena

$$\delta I = \int_{s_A}^{s_B} \left[ \frac{\partial \mathcal{L}}{\partial x^\mu} - \frac{d}{ds} \left( \frac{\partial \mathcal{L}}{\partial \left( \frac{dx^\mu}{ds} \right)} \right) \right] \delta x^\mu ds = 0 \qquad \text{(A.8)}$$

maka didapatkan

$$\frac{\partial \mathcal{L}}{\partial x^\mu} - \frac{d}{ds} \left( \frac{\partial \mathcal{L}}{\partial \left( \frac{dx^\mu}{ds} \right)} \right) = 0 \qquad \text{(A.9)}$$

pers.(A.9) disebut persamaan *Euler-Lagrange*.

Dari persamaan metrik

$$\begin{aligned} dl^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ dl &= (g_{\mu\nu} dx^\mu dx^\nu)^{\frac{1}{2}} \\ \frac{dl}{ds} &= \left( g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \right)^{\frac{1}{2}} \\ dl &= \left( g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \right)^{\frac{1}{2}} ds \qquad \text{(A.10)} \end{aligned}$$

panjang kurva stasioner adalah

$$I = l(s) = \int dl = \int \left( g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \right)^{\frac{1}{2}} ds \qquad \text{(A.11)}$$

pers.(A.11) jika dibandingkan dengan pers.(A.2) akan didapatkan Lagrangian yaitu

$$\mathcal{L} = \left( g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \right)^{\frac{1}{2}} \qquad \text{(A.12)}$$

Dari bentuk Lagrangian pers.(A.12) ini disubstitusikan ke dalam pers.(A.9). Suku ke-2 pers.(A.9) adalah

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x^\lambda} &= \frac{\partial}{\partial x^\lambda}\left(g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\right)^{\frac{1}{2}} \\
&= \frac{\partial\left(g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\right)^{\frac{1}{2}}}{\partial\left(g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\right)}\frac{\partial\left(g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\right)}{\partial x^\lambda} \\
&= \frac{1}{2}\underbrace{\left(g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\right)^{-\frac{1}{2}}}_{=1}\left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\right) \\
&\quad +\frac{1}{2}\underbrace{\left(g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\right)^{-\frac{1}{2}}}_{=1}\left(g_{\mu\nu}\underbrace{\left\{\frac{\partial}{\partial x^\lambda}\frac{dx^\mu}{ds}\right\}}_{=0}\frac{dx^\nu}{ds}\right) \\
&\quad +\frac{1}{2}\underbrace{\left(g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\right)^{-\frac{1}{2}}}_{=1}\left(g_{\mu\nu}\frac{dx^\mu}{ds}\underbrace{\left\{\frac{\partial}{\partial x^\lambda}\frac{dx^\nu}{ds}\right\}}_{=0}\right) \\
&= \frac{1}{2}\left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\right) \qquad \text{(A.13)}
\end{aligned}$$

dan suku pertama adalah:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \left(\frac{dx^\lambda}{ds}\right)} &= \frac{\partial}{\partial \left(\frac{dx^\lambda}{ds}\right)} \left( g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \right)^{\frac{1}{2}} \\
&= \frac{1}{2} \underbrace{\left( g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \right)^{-\frac{1}{2}}}_{=1} \left( \frac{\partial g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}}{\partial \left(\frac{dx^\lambda}{ds}\right)} \right) \\
&= \frac{1}{2} \left\{ \underbrace{\frac{\partial g_{\mu\nu}}{\partial \left(\frac{dx^\lambda}{ds}\right)} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}}_{=0} + g_{\mu\nu} \delta^\mu_\lambda \frac{dx^\nu}{ds} + g_{\mu\nu} \frac{dx^\mu}{ds} \delta^\nu_\lambda \right\} \\
&= \frac{1}{2} \left\{ \underbrace{g_{\lambda\nu} \frac{dx^\nu}{ds}}_{\nu \to \mu} + g_{\mu\lambda} \frac{dx^\mu}{ds} \right\} \\
&= \frac{1}{2} \left\{ g_{\lambda\mu} \frac{dx^\mu}{ds} + g_{\mu\lambda} \frac{dx^\mu}{ds} \right\} \\
&= g_{\mu\lambda} \frac{dx^\mu}{ds} \qquad \text{(A.14)}
\end{aligned}
$$

maka

$$
\begin{aligned}
\frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{dx^\lambda}{ds}\right)}\right) &= \frac{d}{ds}\left(g_{\mu\lambda}\frac{dx^\mu}{ds}\right) \\
&= \frac{dg_{\mu\lambda}}{ds}\frac{dx^\mu}{ds} + g_{\mu\lambda}\frac{d}{ds}\left(\frac{dx^\mu}{ds}\right) \\
&= \frac{\partial g_{\mu\lambda}}{\partial x^\nu}\frac{dx^\nu}{ds}\frac{dx^\mu}{ds} + g_{\mu\lambda}\frac{d^2x^\mu}{ds^2} \\
&= g_{\mu\lambda}\frac{d^2x^\mu}{ds^2} + \frac{1}{2}\left\{\frac{\partial g_{\mu\lambda}}{\partial x^\nu}\frac{dx^\nu}{ds}\frac{dx^\mu}{ds} + \underbrace{\frac{\partial g_{\mu\lambda}}{\partial x^\nu}\frac{dx^\nu}{ds}\frac{dx^\mu}{ds}}_{\nu\to\mu}\right\} \\
&= g_{\mu\lambda}\frac{d^2x^\mu}{ds^2} + \frac{1}{2}\left\{\frac{\partial g_{\mu\lambda}}{\partial x^\nu}\frac{dx^\nu}{ds}\frac{dx^\mu}{ds} + \frac{\partial g_{\nu\lambda}}{\partial x^\mu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\right\} \\
&= g_{\mu\lambda}\frac{d^2x^\mu}{ds^2} + \frac{1}{2}\left\{\frac{\partial g_{\mu\lambda}}{\partial x^\nu} + \frac{\partial g_{\nu\lambda}}{\partial x^\mu}\right\}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} \qquad \text{(A.15)}
\end{aligned}
$$

Dari pers.(A.13) dan pers.(A.15) jika disubstitusikan ke pers.(A.9) sehingga

$$
\begin{aligned}
g_{\rho\lambda}\frac{d^2x^\rho}{ds^2} + \frac{1}{2}\left\{\frac{\partial g_{\mu\lambda}}{\partial x^\nu} + \frac{\partial g_{\nu\lambda}}{\partial x^\mu}\right\}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} - \frac{1}{2}\left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\right) &= 0 \\
g_{\rho\lambda}\frac{d^2x^\rho}{ds^2} + \frac{1}{2}\left\{\frac{\partial g_{\mu\lambda}}{\partial x^\nu} + \frac{\partial g_{\nu\lambda}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda}\right\}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} &= 0 \\
g_{\rho\lambda}\frac{d^2x^\rho}{ds^2} + \Gamma_{\lambda,\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} &= 0 \\
g^{\rho\lambda}\left\{g_{\rho\lambda}\frac{d^2x^\rho}{ds^2} + \Gamma_{\lambda,\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\right\} &= 0 \\
\frac{d^2x^\rho}{ds^2} + \Gamma^\rho_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} &= 0 \\
&\qquad \text{(A.16)}
\end{aligned}
$$

pers.(A.16) adalah persamaan geodesik.

## A.2 Tensor Kurvatur

Perubahan vektor kovarian $A_\mu$

$$\Delta A_\mu = \oint_C \delta A_\mu = \oint_C \Gamma^\rho_{\mu\nu} A_\rho dx^\nu \tag{A.17}$$

Dari teorema Stokes, integral kontur tertutup $C$ dapat ditransformasi ke integral luasan $S$ sehingga

$$\Delta A_\mu = \frac{1}{2} \oint_C \left[ D_\gamma \left( \Gamma^\rho_{\mu\nu} A_\rho \right) - D_\nu \left( \Gamma^\rho_{\mu\gamma} A_\rho \right) \right] dS^{\gamma\nu} \tag{A.18}$$

dengan $D_\gamma$ adalah operator turunan kovarian. Operator ini dapat disimplifikasi menjadi operator turunan biasa non-tensor $D_\gamma \to \partial_\gamma$ seperti berikut:

$$\begin{aligned} D_\gamma \left( \Gamma^\rho_{\mu\nu} A_\rho \right) &\equiv D_\gamma C_{\mu\nu} \\ &= \partial_\gamma C_{\mu\nu} - \Gamma^\sigma_{\mu\gamma} C_{\sigma\nu} - \Gamma^\sigma_{\gamma\nu} C_{\mu\sigma} \end{aligned} \tag{A.19}$$

$$\begin{aligned} D_\nu \left( \Gamma^\rho_{\mu\gamma} A_\rho \right) &\equiv D_\nu C_{\mu\gamma} \\ &= \partial_\nu C_{\mu\gamma} - \Gamma^\sigma_{\mu\nu} C_{\sigma\gamma} - \Gamma^\sigma_{\nu\gamma} C_{\mu\sigma} \end{aligned} \tag{A.20}$$

Dari pers.(A.19) dan pers.(A.20), maka

$$\begin{aligned} D_\gamma \left( \Gamma^\rho_{\mu\nu} A_\rho \right) - D_\nu \left( \Gamma^\rho_{\mu\gamma} A_\rho \right) &= D_\gamma C_{\mu\nu} - D_\nu C_{\mu\gamma} \\ &= \partial_\gamma C_{\mu\nu} - \Gamma^\sigma_{\mu\gamma} C_{\sigma\nu} - \Gamma^\sigma_{\gamma\nu} C_{\mu\sigma} \\ &\quad - \left( \partial_\nu C_{\mu\gamma} - \Gamma^\sigma_{\mu\nu} C_{\sigma\gamma} - \Gamma^\sigma_{\nu\gamma} C_{\mu\sigma} \right) \\ &= \partial_\gamma C_{\mu\nu} - \Gamma^\sigma_{\mu\gamma} C_{\sigma\nu} - \partial_\nu C_{\mu\gamma} + \Gamma^\sigma_{\mu\nu} C_{\sigma\gamma} \end{aligned} \tag{A.21}$$

karena tensor permukaan merupakan tensor simetrik ($dS^{\gamma\nu} = dS^{\nu\gamma}$), maka

$$
\begin{aligned}
(D_\gamma C_{\mu\nu} - D_\nu C_{\mu\gamma})dS^{\gamma\nu} &= \left(\partial_\gamma C_{\mu\nu} - \Gamma^\sigma_{\mu\gamma} C_{\sigma\nu} - \partial_\nu C_{\mu\gamma} + \Gamma^\sigma_{\mu\nu} C_{\sigma\gamma}\right) dS^{\gamma\nu} \\
&= \left(\partial_\gamma C_{\mu\nu} - \partial_\nu C_{\mu\gamma} + \Gamma^\sigma_{\mu\nu} C_{\sigma\gamma} - \underbrace{\Gamma^\sigma_{\mu\gamma} C_{\sigma\nu}}_{\gamma\leftrightarrow\nu}\right) dS^{\gamma\nu} \\
&= \left(\partial_\gamma C_{\mu\nu} - \partial_\nu C_{\mu\gamma} + \Gamma^\sigma_{\mu\nu} C_{\sigma\gamma} - \Gamma^\sigma_{\mu\nu} C_{\sigma\gamma}\right) dS^{\gamma\nu} \\
&= \left(\partial_\gamma C_{\mu\nu} - \partial_\nu C_{\mu\gamma}\right) dS^{\gamma\nu} \\
&= \left(\partial_\gamma \Gamma^\rho_{\mu\nu} A_\rho - \partial_\nu \Gamma^\rho_{\mu\gamma} A_\rho\right) dS^{\gamma\nu} \qquad \text{(A.22)}
\end{aligned}
$$

Dari pers.(A.22), maka simplifikasi $D_\gamma \to \partial\gamma$ bisa digunakan, sehingga pers.(A.18) menjadi:

$$
\begin{aligned}
\Delta A_\mu &= \frac{1}{2}\oint_C \left[D_\gamma \left(\Gamma^\rho_{\mu\nu} A_\rho\right) - D_\nu \left(\Gamma^\rho_{\mu\gamma} A_\rho\right)\right] dS^{\gamma\nu} \\
&= \frac{1}{2}\oint_C \left[\partial_\gamma \left(\Gamma^\rho_{\mu\nu} A_\rho\right) - \partial_\nu \left(\Gamma^\rho_{\mu\nu}\gamma A_\rho\right)\right] dS^{\gamma\nu} \\
&= \frac{1}{2}\oint_C [\partial_\gamma \Gamma^\rho_{\mu\nu} A_\rho + \Gamma^\rho_{\mu\nu} \partial_\gamma A_\rho \\
&\quad -\partial_\nu \Gamma^\rho_{\mu\gamma} A_\rho - \Gamma^\rho_{\mu\gamma} \partial_\nu A_\rho] dS^{\gamma\rho} \qquad \text{(A.23)}
\end{aligned}
$$

karena

$$\delta A_\rho = \Gamma^\lambda_{\rho\gamma} A_\lambda dx^\gamma \to \partial_\gamma A_\rho = \Gamma^\lambda_{\rho\gamma} A_\lambda \qquad \text{(A.24)}$$

$$
\begin{aligned}
\Delta A_\mu &= \frac{1}{2}\oint_C [\partial_\gamma \Gamma^\rho_{\mu\nu} A_\rho + \Gamma^\rho_{\mu\nu}\Gamma^\lambda_{\rho\gamma} A_\lambda - \partial_\nu \Gamma^\rho_{\mu\gamma} A_\rho - \Gamma^\rho_{\mu\gamma}\Gamma^\lambda_{\rho\nu} A_\lambda] dS^{\gamma\rho} \\
&= \frac{1}{2}\oint_C [\partial_\gamma \Gamma^\rho_{\mu\nu} A_\rho - \partial_\nu \Gamma^\rho_{\mu\gamma} A_\rho + \Gamma^\rho_{\mu\nu}\Gamma^\lambda_{\rho\gamma} A_\lambda - \Gamma^\rho_{\mu\gamma}\Gamma^\lambda_{\rho\nu} A_\lambda] dS^{\gamma\rho} \\
&= \frac{1}{2}\oint_C \left[\partial_\gamma \Gamma^\rho_{\mu\nu} A_\rho - \partial_\nu \Gamma^\rho_{\mu\gamma} A_\rho + \underbrace{\Gamma^\rho_{\mu\nu}\Gamma^\lambda_{\rho\gamma} A_\lambda - \Gamma^\rho_{\mu\gamma}\Gamma^\lambda_{\rho\nu} A_\lambda}_{\lambda\leftrightarrow\rho}\right] dS^{\gamma\rho} \\
&= \frac{1}{2}\oint_C \left[\partial_\gamma \Gamma^\rho_{\mu\nu} A_\rho - \partial_\nu \Gamma^\rho_{\mu\gamma} A_\rho + \Gamma^\lambda_{\mu\nu}\Gamma^\rho_{\lambda\gamma} A_\rho - \Gamma^\lambda_{\mu\gamma}\Gamma^\rho_{\lambda\nu} A_\rho\right] dS^{\gamma\lambda} \\
&= \frac{1}{2}\oint_C \left[\partial_\gamma \Gamma^\rho_{\mu\nu} - \partial_\nu \Gamma^\rho_{\mu\gamma} + \Gamma^\lambda_{\mu\nu}\Gamma^\rho_{\lambda\gamma} - \Gamma^\lambda_{\mu\gamma}\Gamma^\rho_{\lambda\nu}\right] A_\rho dS^{\gamma\lambda} \\
&\equiv \frac{1}{2}\oint_C R^\rho_{\mu\gamma\nu} A_\rho dS^{\gamma\lambda} \qquad \text{(A.25)}
\end{aligned}
$$

dengan $R^\rho_{\mu\gamma\nu}$ adalah tensor kurvatur Riemann

$$R^\rho_{\mu\gamma\nu} = \partial_\gamma \Gamma^\rho_{\mu\nu} - \partial_\nu \Gamma^\rho_{\mu\gamma} + \Gamma^\lambda_{\mu\nu}\Gamma^\rho_{\lambda\gamma} - \Gamma^\lambda_{\mu\gamma}\Gamma^\rho_{\lambda\nu} \qquad \text{(A.26)}$$

Tensor Ricci didefinisikan dari tensor kurvatur Riemann

$$R_{\mu\gamma} \equiv R^\rho_{\mu\gamma\rho} = \partial_\gamma \Gamma^\rho_{\mu\rho} - \partial_\rho \Gamma^\rho_{\mu\gamma} + \Gamma^\lambda_{\mu\rho}\Gamma^\rho_{\lambda\gamma} - \Gamma^\lambda_{\mu\gamma}\Gamma^\rho_{\lambda\rho} \qquad \text{(A.27)}$$

serta scalar Ricci adalah

$$R \equiv R^\mu_\mu = g^{\mu\gamma} R_{\mu\gamma} \qquad \text{(A.28)}$$

## A.3 Tensor Energi-Momentum

Diturunkan kembali persamaan *Euler-Lagrange* secara umum untuk rapat Lagrangian $L$ yang bergantung kuantitas $q$ dan turunan pertamanya

$q,_\mu$ yaitu

$$
\begin{aligned}
I &= \int \mathcal{L}(q, q,_\mu)dV dt \\
&= \frac{1}{c}\int \mathcal{L}(q, q,_\mu)d\Omega \\
\delta I &= \frac{1}{c}\int \delta\mathcal{L}(q, q,_\mu)d\Omega \\
&= \frac{1}{c}\int \left(\frac{\partial \mathcal{L}}{\partial q}\delta q + \frac{\partial \mathcal{L}}{\partial q,_\mu}\delta q,_\mu\right) d\Omega \\
&= \frac{1}{c}\int \left(\frac{\partial \mathcal{L}}{\partial q}\delta q + \frac{\partial \mathcal{L}}{\partial(\partial_\mu q)}\delta \frac{\partial q}{\partial x^\mu}\right) d\Omega \\
&= \frac{1}{c}\int \left(\frac{\partial \mathcal{L}}{\partial q}\delta q + \frac{\partial \mathcal{L}}{\partial(\partial_\mu q)}\frac{\partial}{\partial x^\mu}\delta q\right) d\Omega \\
&= \frac{1}{c}\int \left(\frac{\partial \mathcal{L}}{\partial q}\delta q + \frac{\partial}{\partial x^\mu}\left(\delta q\frac{\partial \mathcal{L}}{\partial(\partial_\mu q)}\right) - \left(\frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial(\partial_\mu q)}\right)\delta q\right) d\Omega \\
&= \frac{1}{c}\int \left(\frac{\partial \mathcal{L}}{\partial q}\delta q + \underbrace{\partial_\mu\left\{\frac{\partial \mathcal{L}}{\partial(\partial_\mu q)}\delta q\right\}}_{=0(\text{teorema } \textit{Gauss})} - \left\{\partial_\mu\frac{\partial \mathcal{L}}{\partial(\partial_\mu q)}\right\}\delta q\right) d\Omega \\
&= \frac{1}{c}\int \left(\frac{\partial \mathcal{L}}{\partial q}\delta q - \partial_\mu\frac{\partial \mathcal{L}}{\partial(\partial_\mu q)}\delta q\right) d\Omega \\
&= \frac{1}{c}\int \left(\frac{\partial \mathcal{L}}{\partial q} - \partial_\mu\frac{\partial \mathcal{L}}{\partial(\partial_\mu q)}\right) \delta q d\Omega \\
&= 0
\end{aligned}
\tag{A.29}
$$

sehingga didapatkan persamaan *Euler-Lagrange*:

$$
\frac{\partial \mathcal{L}}{\partial q} - \partial_\mu\frac{\partial \mathcal{L}}{\partial(\partial_\mu q)} = 0 \tag{A.30}
$$

Turunan pertama terhadap Lagrangian

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial x^\mu} &= \frac{\partial\mathcal{L}}{\partial q}\frac{\partial q}{\partial x^\mu} + \frac{\partial\mathcal{L}}{\partial(\partial_\nu q)}\frac{\partial(\partial_\nu q)}{\partial x^\mu} \\
\frac{\partial\mathcal{L}}{\partial x^\mu} &= \partial_\gamma\frac{\partial\mathcal{L}}{\partial(\partial_\gamma q)}\frac{\partial q}{\partial x^\mu} + \frac{\partial\mathcal{L}}{\partial(\partial_\nu q)}\frac{\partial(\partial_\nu q)}{\partial x^\mu} \\
\frac{\partial\mathcal{L}}{\partial x^\mu} &= \partial_\gamma\frac{\partial\mathcal{L}}{\partial(\partial_\gamma q)}\partial_\mu q + \frac{\partial\mathcal{L}}{\partial(\partial_\nu q)}\partial_\mu\partial_\nu q \\
\frac{\partial\mathcal{L}}{\partial x^\mu} &= \underbrace{\partial_\gamma\left\{\partial_\mu q\frac{\partial\mathcal{L}}{\partial(\partial_\gamma q)}\right\} - \frac{\partial\mathcal{L}}{\partial(\partial_\gamma q)}\partial_\gamma\partial_\mu q}_{\gamma\to\nu} + \frac{\partial\mathcal{L}}{\partial(\partial_\nu q)}\partial_\mu\partial_\nu q \\
\frac{\partial\mathcal{L}}{\partial x^\mu} &= \partial_\nu\left\{\partial_\mu q\frac{\partial\mathcal{L}}{\partial(\partial_\nu q)}\right\} \\
\frac{\partial\mathcal{L}}{\partial x^\nu}\frac{\partial x^\nu}{\partial x^\mu} &= \partial_\nu\left\{\partial_\mu q\frac{\partial\mathcal{L}}{\partial(\partial_\nu q)}\right\} \\
\frac{\partial\mathcal{L}}{\partial x^\nu}\delta^\nu_\mu &= \partial_\nu\left\{\partial_\mu q\frac{\partial\mathcal{L}}{\partial(\partial_\nu q)}\right\} \\
\frac{\partial\mathcal{L}}{\partial x^\nu}\delta^\nu_\mu &= \partial_\nu\left\{\partial_\mu q\frac{\partial\mathcal{L}}{\partial(\partial_\nu q)}\right\} \\
0 &= \partial_\nu\left\{\partial_\mu q\frac{\partial\mathcal{L}}{\partial(\partial_\nu q)} - \delta^\nu_\mu\mathcal{L}\right\} \equiv \partial_\nu T^\nu_\mu \qquad \text{(A.31)}
\end{aligned}$$

maka

$$T^\nu_\mu = \partial_\mu q\frac{\partial\mathcal{L}}{\partial(\partial_\nu q)} - \delta^\nu_\mu\mathcal{L} \qquad \text{(A.32)}$$

Pers.(A.32) adalah tensor energi-momentum.

Untuk Medan Elektromagnetik, rapat Lagrangian adalah

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu J^\mu \qquad \text{(A.33)}$$

karena $F = F(\partial_\mu A_\mu) \longrightarrow \mathcal{L} = \mathcal{L}(A_\mu, \partial_\mu A_\mu)$, maka pers.(A.30) menjadi

$$\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = 0 \tag{A.34}$$

Substitusi pers.(A.33) ke dalam pers.(A.34). Suku pertama pers.(A.34) adalah

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial A_\nu} &= -\frac{\partial A_\mu J^\mu}{\partial A_\nu} \\ &= -\delta^\nu_\mu J^\mu \\ &= -J^\nu \end{aligned} \tag{A.35}$$

untuk suku ke-2:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} &= -\frac{1}{4}\frac{\partial}{\partial(\partial_\mu A_\nu)}\left\{F_{\alpha\beta}F^{\alpha\beta}\right\} \\ &= -\frac{1}{4}g^{\rho\alpha}g^{\sigma\beta}\frac{\partial}{\partial(\partial_\mu A_\nu)}\left\{F_{\alpha\beta}F_{\rho\sigma}\right\} \\ &= -\frac{1}{4}g^{\rho\alpha}g^{\sigma\beta}\left\{\frac{\partial\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right)}{\partial(\partial_\mu A_\nu)}F_{\rho\sigma} + F_{\alpha\beta}\frac{\partial\left(\partial_\rho A_\sigma - \partial_\sigma A_\rho\right)}{\partial(\partial_\mu A_\nu)}\right\} \\ &= -\frac{1}{4}g^{\rho\alpha}g^{\sigma\beta}\left\{\left(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\mu_\beta\delta^\nu_\alpha\right)F_{\rho\sigma} + F_{\alpha\beta}\left(\delta^\mu_\rho\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\nu_\rho\right)\right\} \\ &= -\frac{1}{4}\left\{\left(g^{\rho\mu}g^{\sigma\nu} - g^{\rho\nu}g^{\sigma\mu}\right)F_{\rho\sigma} + F_{\alpha\beta}\left(g^{\mu\alpha}g^{\nu\beta} - g^{\nu\alpha}g^{\mu\beta}\right)\right\} \\ &= -\frac{1}{4}\left\{F^{\mu\nu} - F^{\nu\mu} + F^{\mu\nu} - F^{\nu\mu}\right\} \\ &= -\frac{1}{4}\left\{F^{\mu\nu} + F^{\mu\nu} + F^{\mu\nu} + F^{\mu\nu}\right\} \\ &= -\frac{1}{4}4F^{\mu\nu} \\ &= -F^{\mu\nu} \end{aligned} \tag{A.36}$$

Substitusi pers.(A.35) dan pers.(A.36) ke dalam pers.(A.34) sehingga didapatkan

$$\partial_\mu F^{\mu\nu} = J^\nu \tag{A.37}$$

Pers.(A.37) adalah persamaan Maxwell. Untuk keadaan tidak ada arus $J^\nu = 0$, maka

$$\partial_\mu F^{\mu\nu} = 0 \tag{A.38}$$

Parameter $q$ untuk medan elektromagnetik adalah $A_\mu$, sehingga tensor energi-momentum pers.(A.32) untuk medan elektromagnetik adalah

$$\begin{aligned}
T^\nu_\mu &= \partial_\mu A_\lambda \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\lambda)} - \delta^\nu_\mu \mathcal{L} \\
&= -F^{\nu\lambda}\partial_\mu A_\lambda + \frac{1}{4}\delta^\nu_\mu F^{\lambda\gamma}F_{\lambda\gamma} \\
&= -F^{\nu\lambda}\left(\partial_\mu A_\lambda - \partial_\lambda A_\mu\right) - F^{\nu\lambda}\partial_\lambda A_\mu + \frac{1}{4}\delta^\nu_\mu F^{\lambda\gamma}F_{\lambda\gamma} \\
&= -F^{\nu\lambda}F_{\mu\lambda} - F^{\nu\lambda}\partial_\lambda A_\mu + \frac{1}{4}\delta^\nu_\mu F^{\lambda\gamma}F_{\lambda\gamma} \\
&= -F^{\nu\lambda}F_{\mu\lambda} - \partial_\lambda(F^{\nu\lambda}A_\mu) + \underbrace{A_\mu\partial_\lambda F^{\nu\lambda}}_{=0} + \frac{1}{4}\delta^\nu_\mu F^{\lambda\gamma}F_{\lambda\gamma} \\
&= -F^{\nu\lambda}F_{\mu\lambda} - \partial_\lambda(F^{\nu\lambda}A_\mu) + \frac{1}{4}\delta^\nu_\mu F^{\lambda\gamma}F_{\lambda\gamma}
\end{aligned} \tag{A.39}$$

Karena $\partial_\nu T^\nu_\mu = 0$, maka suku ke-2:

$$\begin{aligned}
\partial_\nu\partial_\lambda(F^{\nu\lambda}A_\mu) &= \frac{1}{2}\left(\partial_\nu\partial_\lambda(F^{\nu\lambda}A_\mu) + \underbrace{\partial_\nu\partial_\lambda(F^{\nu\lambda}A_\mu)}_{\nu\leftrightarrow\lambda}\right) \\
&= \frac{1}{2}\left\{\partial_\nu\partial_\lambda(F^{\nu\lambda}A_\mu) + \partial_\lambda\partial_\nu(F^{\lambda\nu}A_\mu)\right\} \\
&= \frac{1}{2}\left\{\partial_\nu\partial_\lambda(F^{\nu\lambda}A_\mu) - \partial_\nu\partial_\lambda(F^{\nu\lambda}A_\mu)\right\} \\
&= 0
\end{aligned} \tag{A.40}$$

Sehingga pers.(A.39) menjadi

$$T^\nu_\mu = -F^{\nu\lambda}F_{\mu\lambda} + \frac{1}{4}\delta^\nu_\mu F^{\lambda\gamma}F_{\lambda\gamma} \tag{A.41}$$

## A.4 Persamaan Medan Einstein

Aksi medan gravitasi pada ruang vakum

$$I_G = \frac{1}{c} \int_\Omega \mathcal{L}_G \left(g_{\mu\nu}, \partial_\sigma g_{\mu\nu}\right) \sqrt{-g} d\Omega \tag{A.42}$$

dengan bentuk dari $\mathcal{L}_G$ adalah

$$\mathcal{L}_G = -\frac{c^4}{16\pi G} R \tag{A.43}$$

Pers.(A.43)disubstitusikan ke pers.(A.42)

$$I_G = -\frac{c^3}{16\pi G} \int_\Omega R\sqrt{-g} d\Omega \tag{A.44}$$

jika dilakukan variasi terhadap $I_G$ di atas, maka

$$\delta I_G = -\frac{c^3}{16\pi G} \int_\Omega \delta\sqrt{-g}R \, d\Omega \tag{A.45}$$

dengan

$$\begin{aligned} \delta\left(\sqrt{-g}R\right) &= \delta\left(\sqrt{-g}g^{\mu\nu}R_{\mu\nu}\right) \\ &= \left(\delta\sqrt{-g}\right)g^{\mu\nu}R_{\mu\nu} + \sqrt{-g}\left(\delta g^{\mu\nu}\right)R_{\mu\nu} \\ &\quad +\sqrt{-g}g^{\mu\nu}\left(\delta R_{\mu\nu}\right) \end{aligned} \tag{A.46}$$

karena

$$\delta g = g g^{\mu\nu}\delta g_{\mu\nu} = -g g_{\mu\nu}\delta g^{\mu\nu} \tag{A.47}$$

maka

$$
\begin{aligned}
\delta\sqrt{-g} &= \frac{\partial\sqrt{-g}}{\partial g}\delta g \\
&= -\frac{1}{2\sqrt{-g}}\delta g \\
&= -\frac{1}{2\sqrt{-g}}\left(-g g_{\mu\nu}\delta g^{\mu\nu}\right) \\
&= -\frac{\sqrt{-g}}{2}g_{\mu\nu}\delta g^{\mu\nu} \qquad \text{(A.48)}
\end{aligned}
$$

Substitusi pers.(A.48) ke pers.(A.46)

$$
\begin{aligned}
\delta\left(\sqrt{-g}R\right) &= -\frac{\sqrt{-g}}{2}g_{\mu\nu}\delta g^{\mu\nu}g^{\mu\nu}R_{\mu\nu} + \sqrt{-g}\left(\delta g^{\mu\nu}\right)R_{\mu\nu} \\
&\quad +\sqrt{-g}g^{\mu\nu}\left(\delta R_{\mu\nu}\right) \\
&= -\frac{\sqrt{-g}}{2}g_{\mu\nu}\delta g^{\mu\nu}R + \sqrt{-g}\left(\delta g^{\mu\nu}\right)R_{\mu\nu} \\
&\quad +\sqrt{-g}g^{\mu\nu}\left(\delta R_{\mu\nu}\right) \\
&= \sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu} + \sqrt{-g}g^{\mu\nu}\left(\delta R_{\mu\nu}\right) \\
&\qquad \text{(A.49)}
\end{aligned}
$$

dengan suku ke-3 adalah

$$
\begin{aligned}
g^{\mu\nu}\delta R_{\mu\nu} &= g^{\mu\nu}\delta\left(\partial_\nu\Gamma^\rho_{\mu\rho} - \partial_\rho\Gamma^\rho_{\mu\nu} + \Gamma^\lambda_{\mu\rho}\Gamma^\rho_{\lambda\nu} - \Gamma^\lambda_{\mu\nu}\Gamma^\rho_{\lambda\rho}\right) \\
&= g^{\mu\nu}\delta\left(\partial_\nu\Gamma^\rho_{\mu\rho} - \partial_\rho\Gamma^\rho_{\mu\nu}\right) \\
&\quad + g^{\mu\nu}\delta\Gamma^\lambda_{\mu\rho}\Gamma^\rho_{\lambda\nu} + g^{\mu\nu}\Gamma^\lambda_{\mu\rho}\delta\Gamma^\rho_{\lambda\nu} \\
&\quad - g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu}\Gamma^\rho_{\lambda\rho} - g^{\mu\nu}\Gamma^\lambda_{\mu\nu}\delta\Gamma^\rho_{\lambda\rho} \\
&\quad + \underbrace{g^{\mu\nu}\Gamma^\rho_{\nu\rho}\delta\Gamma^\lambda_{\mu\lambda} - g^{\mu\nu}\Gamma^\rho_{\nu\rho}\delta\Gamma^\lambda_{\mu\lambda}}_{\text{suku tambahan=0}} \\
&= g^{\mu\nu}\delta\underbrace{\left(\partial_\nu\Gamma^\rho_{\mu\rho} - \partial_\rho\Gamma^\rho_{\mu\nu}\right)}_{\rho\to\lambda} \\
&\quad + g^{\mu\nu}\underbrace{\left(\Gamma^\rho_{\nu\rho}\delta\Gamma^\lambda_{\mu\lambda} - \Gamma^\rho_{\lambda\rho}\delta\Gamma^\lambda_{\mu\nu}\right)}_{\rho\to\gamma} \\
&\quad + \underbrace{g^{\mu\nu}\delta\Gamma^\lambda_{\mu\rho}\Gamma^\rho_{\lambda\nu}}_{\lambda\to\mu,\mu\to\rho,\rho\to\lambda} + \underbrace{g^{\mu\nu}\Gamma^\lambda_{\mu\rho}\delta\Gamma^\rho_{\lambda\nu}}_{\rho\leftrightarrow\nu} \\
&\quad \underbrace{-g^{\mu\nu}\Gamma^\lambda_{\mu\nu}\delta\Gamma^\rho_{\lambda\rho}}_{\lambda\to\mu,\mu\to\rho,\rho\to\lambda} - \underbrace{g^{\mu\nu}\Gamma^\rho_{\nu\rho}\delta\Gamma^\lambda_{\mu\lambda}}_{\nu\leftrightarrow\rho} \\
&= g^{\mu\nu}\delta\left(\partial_\nu\Gamma^\lambda_{\mu\lambda} - \partial_\lambda\Gamma^\lambda_{\mu\nu}\right) \\
&\quad + g^{\mu\nu}\left(\Gamma^\gamma_{\nu\gamma}\delta\Gamma^\lambda_{\mu\lambda} - \Gamma^\gamma_{\lambda\gamma}\delta\Gamma^\lambda_{\mu\nu}\right) \\
&\quad + g^{\rho\nu}\Gamma^\mu_{\rho\lambda}\delta\Gamma^\lambda_{\nu\mu} + g^{\mu\rho}\Gamma^\nu_{\rho\lambda}\delta\Gamma^\lambda_{\mu\nu} \\
&\quad - g^{\rho\nu}\Gamma^\mu_{\rho\nu}\delta\Gamma^\lambda_{\mu\lambda} - g^{\mu\rho}\Gamma^\nu_{\rho\nu}\delta\Gamma^\lambda_{\mu\lambda} \\
&= g^{\mu\nu}\delta\left(\partial_\nu\Gamma^\lambda_{\mu\lambda} - \partial_\lambda\Gamma^\lambda_{\mu\nu}\right) \\
&\quad + g^{\mu\nu}\left(\Gamma^\gamma_{\nu\gamma}\delta\Gamma^\lambda_{\mu\lambda} - \Gamma^\gamma_{\lambda\gamma}\delta\Gamma^\lambda_{\mu\nu}\right) \\
&\quad - \left(-g^{\rho\nu}\Gamma^\mu_{\rho\lambda} - g^{\mu\rho}\Gamma^\nu_{\rho\lambda}\right)\delta\Gamma^\lambda_{\mu\nu} \\
&\quad + \left(-g^{\rho\nu}\Gamma^\mu_{\rho\nu} - g^{\mu\rho}\Gamma^\nu_{\rho\nu}\right)\delta\Gamma^\lambda_{\mu\lambda} \qquad \text{(A.50)}
\end{aligned}
$$

Karena turunan kovarian tensor metrik adalah nol

$$\begin{aligned} D_\lambda g^{\mu\nu} &= \partial_\lambda g^{\mu\nu} + \Gamma^\mu_{\rho\lambda} g^{\rho\nu} + \Gamma^\nu_{\rho\lambda} g^{\mu\rho} = 0 \\ \partial_\lambda g^{\mu\nu} &= -\Gamma^\mu_{\rho\lambda} g^{\rho\nu} - \Gamma^\nu_{\rho\lambda} g^{\mu\rho} \end{aligned} \tag{A.51}$$

$$\begin{aligned} D_\nu g^{\mu\nu} &= \partial_\nu g^{\mu\nu} + \Gamma^\mu_{\rho\nu} g^{\rho\nu} + \Gamma^\nu_{\rho\nu} g^{\mu\rho} = 0 \\ \partial_\nu g^{\mu\nu} &= -\Gamma^\mu_{\rho\nu} g^{\rho\nu} - \Gamma^\nu_{\rho\nu} g^{\mu\rho} \end{aligned} \tag{A.52}$$

serta dengan menubstitusikan pers.(A.51) dan pers.(A.52) ke dalam pers.(A.50), maka didapatkan

$$\begin{aligned} g^{\mu\nu}\delta R_{\mu\nu} &= g^{\mu\nu}\delta\left(\partial_\nu \Gamma^\lambda_{\mu\lambda}\right) - g^{\mu\nu}\delta\left(\partial_\lambda \Gamma^\lambda_{\mu\nu}\right) \\ &\quad -\partial_\lambda g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu} + \partial_\nu g^{\mu\nu}\delta\Gamma^\lambda_{\mu\lambda} \\ &\quad +g^{\mu\nu}\left(\Gamma^\gamma_{\nu\gamma}\delta\Gamma^\lambda_{\mu\lambda} - \Gamma^\gamma_{\lambda\gamma}\delta\Gamma^\lambda_{\mu\nu}\right) \\ &= g^{\mu\nu}\partial_\nu\left(\delta\Gamma^\lambda_{\mu\lambda}\right) - g^{\mu\nu}\partial_\lambda\left(\delta\Gamma^\lambda_{\mu\nu}\right) \\ &\quad -\partial_\lambda g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu} + \partial_\nu g^{\mu\nu}\delta\Gamma^\lambda_{\mu\lambda} \\ &\quad +g^{\mu\nu}\left(\Gamma^\gamma_{\nu\gamma}\delta\Gamma^\lambda_{\mu\lambda} - \Gamma^\gamma_{\lambda\gamma}\delta\Gamma^\lambda_{\mu\nu}\right) \\ &= \underbrace{\partial_\nu\left(g^{\mu\nu}\delta\Gamma^\lambda_{\mu\lambda}\right)}_{\lambda\leftrightarrow\nu} - \partial_\lambda\left(g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu}\right) \\ &\quad + \underbrace{\Gamma^\gamma_{\nu\gamma}\left(g^{\mu\nu}\delta\Gamma^\lambda_{\mu\lambda}\right)}_{\lambda\leftrightarrow\nu} - \Gamma^\gamma_{\lambda\gamma}\left(g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu}\right) \\ &= \partial_\lambda\left(g^{\mu\lambda}\delta\Gamma^\nu_{\mu\nu} - g^{\mu\nu}\delta\Gamma^\lambda_{\mu\lambda}\right) \\ &\quad +\Gamma^\gamma_{\lambda\gamma}\left(g^{\mu\lambda}\delta\Gamma^\nu_{\mu\nu} - g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu}\right) \end{aligned} \tag{A.53}$$

Didefinisikan vektor-4

$$\omega^\lambda = g^{\mu\lambda}\delta\Gamma^\nu_{\mu\nu} - g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu} \tag{A.54}$$

Nilai simbol Christoffel

$$\Gamma^\gamma_{\lambda\gamma} = \partial_\lambda \ln\sqrt{-g} \tag{A.55}$$

Kemudian dari definisi vektor-4 dan nilai simbol Christoffel di atas, pers.(A.54) dan pers.(A.55) disubstitusikan ke dalam pers.(A.53) sehingga didapatkan

$$\begin{aligned} g^{\mu\nu}\delta R_{\mu\nu} &= \partial_\lambda \omega^\lambda + \Gamma^\gamma_{\lambda\gamma}\omega^\lambda \\ &= \partial_\lambda \omega^\lambda + \frac{1}{\sqrt{-g}}\partial_\lambda \ln\sqrt{-g}\omega^\lambda \\ &= \frac{1}{\sqrt{-g}}\partial_\lambda\left(\sqrt{-g}\omega^\lambda\right) \qquad \text{(A.56)} \end{aligned}$$

substitusi pers.(A.56) ke dalam pers.(A.49)

$$\begin{aligned} \delta\left(\sqrt{-g}R\right) &= \sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu} + \sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} \\ &= \sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu} + \sqrt{-g}\frac{1}{\sqrt{-g}}\partial_\lambda\left(\sqrt{-g}\omega^\lambda\right) \\ &= \sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu} + \partial_\lambda\left(\sqrt{-g}\omega^\lambda\right) \qquad \text{(A.57)} \end{aligned}$$

pers.(A.57) disubstitusikan pers.(A.45) sehingga

$$\begin{aligned} \delta I_G &= -\frac{c^3}{16\pi G}\int_\Omega \left\{\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu} + \underbrace{\partial_\lambda\left(\sqrt{-g}\omega^\lambda\right)}_{=0(\text{teorema } Gauss)}\right\} d\Omega \\ &= -\frac{c^3}{16\pi G}\int_\Omega \sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu} d\Omega \qquad \text{(A.58)} \end{aligned}$$

Sedangkan aksi oleh massa sumber adalah

$$\delta I_M = \frac{1}{c}\int_\Omega \partial(\sqrt{-g}\mathcal{L}_M)d\Omega \qquad \text{(A.59)}$$

dengan

$$
\begin{aligned}
\delta(\sqrt{-g}\mathcal{L}_M) &= \frac{\partial(\sqrt{-g}\mathcal{L}_M)}{\partial g_{\mu\nu}}\delta g_{\mu\nu} \\
&= \left(\frac{\partial\sqrt{-g}}{\partial g_{\mu\nu}}\mathcal{L}_M + \sqrt{-g}\frac{\partial\mathcal{L}_M}{\partial g_{\mu\nu}}\right)\delta g_{\mu\nu} \\
&= \left(\frac{\partial\sqrt{-g}}{\partial g}\frac{\partial g}{\partial g_{\mu\nu}}\mathcal{L}_M + \sqrt{-g}\frac{\partial\mathcal{L}_M}{\partial g_{\mu\nu}}\right)\delta g_{\mu\nu} \\
&= \left(-\frac{1}{2\sqrt{-g}}\frac{gg^{\mu\nu}\partial g_{\mu\nu}}{\partial g_{\mu\nu}}\mathcal{L}_M + \sqrt{-g}\frac{\partial\mathcal{L}_M}{\partial g_{\mu\nu}}\right)\delta g_{\mu\nu} \\
&= \left(-\frac{1}{2\sqrt{-g}}(-\sqrt{-g}\sqrt{-g})g^{\mu\nu}\mathcal{L}_M + \sqrt{-g}\frac{\partial\mathcal{L}_M}{\partial g_{\mu\nu}}\right)\delta g_{\mu\nu} \\
&= \left(\frac{1}{2}(\sqrt{-g})g^{\mu\nu}\mathcal{L}_M + \sqrt{-g}\frac{\partial\mathcal{L}_M}{\partial g_{\mu\nu}}\right)\delta g_{\mu\nu} \\
&= \left(\frac{1}{2}g^{\mu\nu}\mathcal{L}_M + \frac{\partial\mathcal{L}_M}{\partial g_{\mu\nu}}\right)\sqrt{-g}\delta g_{\mu\nu}
\end{aligned}
\tag{A.60}
$$

karena

$$
T^{\mu\nu} = 2\frac{\partial\mathcal{L}_M}{\partial g^{\mu\nu}} + g^{\mu\nu}\mathcal{L}_M, \tag{A.61}
$$

maka

$$\begin{aligned}
\delta(\sqrt{-g}\mathcal{L}_M) &= \frac{1}{2}\left(g^{\mu\nu}\mathcal{L}_M + 2\frac{\partial \mathcal{L}_M}{\partial g_{\mu\nu}}\right)\sqrt{-g}\delta g_{\mu\nu} \\
&= \frac{1}{2}T^{\mu\nu}\sqrt{-g}\delta g_{\mu\nu} \\
&= \frac{1}{2}\sqrt{-g}T^{\mu\nu}\delta g_{\mu\nu} \\
&= \frac{1}{2}\sqrt{-g}g^{\gamma\mu}g^{\lambda\nu}T_{\gamma\lambda}\delta g_{\mu\nu} \\
&= \frac{1}{2}\sqrt{-g}T_{\gamma\lambda}g^{\gamma\mu}g^{\lambda\nu}\delta g_{\mu\nu} \\
&= \frac{1}{2}\sqrt{-g}T_{\gamma\lambda}\delta g^{\gamma\lambda} \\
&= \frac{1}{2}\sqrt{-g}T_{\mu\nu}\delta g^{\mu\nu}
\end{aligned} \tag{A.62}$$

Pers.(A.62) disubstitusikan ke pers.(A.58) sehingga didapatkan

$$\delta I_M = \frac{1}{2c}\int_\Omega \sqrt{-g}T_{\mu\nu}\delta g^{\mu\nu} \tag{A.63}$$

Aksi total adalah

$$\begin{aligned}
I &= I_G + I_M \\
\delta I &= \delta I_G + \delta I_M = 0
\end{aligned}$$

$$\delta I_G = -\delta I_M \tag{A.64}$$

dari pers.(A.58) dan pers.(A.63), didapatkan

$$\begin{aligned}
\delta I_G &= -\delta I_M \\
-\frac{c^3}{16\pi G}\int_\Omega \sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu} d\Omega &= -\frac{1}{2c}\int_\Omega \sqrt{-g}T_{\mu\nu}\delta g^{\mu\nu} \\
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R &= \frac{8\pi G}{c^4}T_{\mu\nu}
\end{aligned} \tag{A.65}$$

Pers.(A.65) adalah persamaan medan Einstein. Dalam ungkapan tensor campuran, pers.(A.65) menjadi

$$R^{\mu}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu} R = \frac{8\pi G}{c^4} T^{\mu}_{\nu} \quad \text{(A.66)}$$

Jika dilakukan $\nu \to \mu$ pada semua suku pers.(A.66) di atas, maka

$$\begin{aligned} R^{\mu}_{\mu} - \frac{1}{2}\delta^{\mu}_{\mu} R &= \frac{8\pi G}{c^4} T^{\mu}_{\mu} \\ R - \frac{1}{2}\left(\delta^0_0 + \delta^1_1 + \delta^2_2 + \delta^3_3\right) R &= \frac{8\pi G}{c^4} T \\ R - 2R &= \frac{8\pi G}{c^4} T \\ R &= -\frac{8\pi G}{c^4} T \end{aligned} \quad \text{(A.67)}$$

Pers.(A.67) disubstitusikan ke dalam pers.(A.98) sehingga didapatkan ungkapan lain dari persamaan medan Einstein

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right) \quad \text{(A.68)}$$

## A.5 Solusi Schwarzschild

Metrik dengan adanya sumber massa $M$ pada koordinat bola:

$$ds^2 = -U(r)dt^2 + V(r)dr^2 + W(r)r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad \text{(A.69)}$$

Misal diambil $Wr^2 = \hat{r}^2 \longrightarrow \hat{r} = \sqrt{W}r$ sehingga

$$\frac{d\hat{r}}{dr} = \sqrt{W}\left(1 + \frac{r}{2W}\frac{dW}{dr}\right) \quad \text{(A.70)}$$

$$V\, dr^2 = \frac{V}{W}\left(1 + \frac{r}{2W}\frac{dW}{dr}\right)^{-2} d\hat{r}^2 \equiv \hat{V} d\hat{r}^2 \quad \text{(A.71)}$$

sehingga $V \equiv \hat{V}$. Dengan cara yang sama maka bisa didapatkan $U \equiv \hat{U}$. Dengan mengganti $r$ menjadi $\hat{r}$, maka elemen garis diatas akan menjadi

$$ds^2 = -\hat{V}(\hat{r})dt^2 + \hat{U}(\hat{r})dr^2 + \hat{r}^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{A.72}$$

Dengan menghilangkan tanda topi pada persamaan di atas, maka

$$ds^2 = -V(r)dt^2 + U(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{A.73}$$

serta dipilih fungsi dari $U$ dan $V$ adalah

$$U(r) = e^{2\nu(r)}, \text{dan } V(r) = e^{2\lambda(r)} \tag{A.74}$$

maka elemen garisnya akan menjadi

$$ds^2 = -e^{2\nu}c^2dt^2 + e^{2\lambda}dr^2 + (r^2d\theta^2 + r^2\sin^2\theta d\phi^2) \tag{A.75}$$

Maka tensor metriknya

$$g_{\mu\nu} = \begin{pmatrix} -e^{2\nu} & 0 & 0 & 0 \\ 0 & e^{2\lambda} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{A.76}$$

karena

$$g^{\mu\nu} = (g_{\mu\nu})^{-1} \tag{A.77}$$

Maka bentuk kontravarian dari tensor metrik diatas adalah

$$g^{\mu\nu} = \begin{pmatrix} -e^{-2\nu} & 0 & 0 & 0 \\ 0 & e^{-2}\lambda & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^{-2}\sin^{-2}\theta \end{pmatrix} \tag{A.78}$$

Persamaan untuk mencari simbol Christoffel adalah

$$\Gamma_{\mu,\upsilon\rho} = \frac{1}{2}\left(\frac{\partial g_{\mu\upsilon}}{\partial x^{\rho}} + \frac{\partial g_{\rho\mu}}{\partial x^{\upsilon}} - \frac{\partial g_{\upsilon\rho}}{\partial x^{\mu}}\right) \tag{A.79}$$

Maka untuk tensor metrik diatas, komponen-komponennya adalah

$$\begin{aligned}
\Gamma_{0,00} &= \frac{1}{2}\left(\partial_0 g_{00} + \partial_0 g_{00} - \partial_0 g_{00}\right) \\
&= 0 \\
\Gamma_{0,01} = \Gamma_{0,10} &= \frac{1}{2}\left(\partial_1 g_{00} + \partial_0 g_{00} - \partial_0 g_{00}\right) \\
&= \frac{1}{2}\left(\frac{\partial\left(-e^{2\nu}\right)}{\partial r}\right) \\
&= -\nu' e^{2\nu} \\
\Gamma_{0,02} &= \Gamma_{0,20} = \frac{1}{2}\left(\partial_2 g_{00} + \partial_0 g_{00} - \partial_0 g_{00}\right) \\
&= 0 \\
\Gamma_{0,03} = \Gamma_{0,30} &= \frac{1}{2}\left(\partial_3 g_{00} + \partial_0 g_{00} - \partial_0 g_{00}\right) \\
&= 0 \\
\Gamma_{0,11} &= \frac{1}{2}\left(\partial_1 g_{01} + \partial_1 g_{10} - \partial_0 g_{11}\right) \\
&= \frac{1}{2}\left(0 + 0 - 0\right) \\
&= 0 \\
\Gamma_{0,12} = \Gamma_{0,21} &= \frac{1}{2}\left(\partial_2 g_{01} + \partial_1 g_{20} - \partial_0 g_{12}\right) \\
&= \frac{1}{2}\left(0 + 0 - 0\right) \\
&= 0 \\
\Gamma_{0,13} = \Gamma_{0,31} &= \frac{1}{2}\left(\partial_3 g_{01} + \partial_1 g_{30} - \partial_0 g_{13}\right) \\
&= \frac{1}{2}\left(0 + 0 - 0\right) \\
&= 0
\end{aligned}$$

$$
\begin{aligned}
\Gamma_{0,22} &= \frac{1}{2}\left(\partial_2 g_{02} + \partial_2 g_{20} - \partial_0 g_{22}\right) \\
&= 0 \\
\Gamma_{0,23} = \Gamma_{0,32} &= \frac{1}{2}\left(\partial_3 g_{02} + \partial_2 g_{03} - \partial_0 g_{23}\right) \\
&= 0 \\
\Gamma_{0,33} &= \frac{1}{2}\left(\partial_3 g_{03} + \partial_3 g_{30} - \partial_0 g_{33}\right) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_{1,00} &= \frac{1}{2}\left(\partial_0 g_{10} + \partial_0 g_{01} - \partial_1 g_{00}\right) \\
&= \frac{1}{2}\left(0 + 0 - \frac{\partial\left(-e^{2\nu}\right)}{\partial r}\right) \\
&= \nu' e^{2\nu} \\
\Gamma_{1,01} = \Gamma_{1,10} &= \frac{1}{2}\left(\partial_1 g_{10} + \partial_0 g_{11} - \partial_1 g_{01}\right) \\
&= 0 \\
\Gamma_{1,02} = \Gamma_{1,20} &= \frac{1}{2}\left(\partial_2 g_{10} + \partial_0 g_{21} - \partial_1 g_{02}\right) \\
&= 0 \\
\Gamma_{1,03} = \Gamma_{1,30} &= \frac{1}{2}\left(\partial_3 g_{10} + \partial_0 g_{31} - \partial_1 g_{03}\right) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_{1,11} &= \frac{1}{2}\left(\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11}\right) \\
&= \frac{1}{2}\left(2\lambda' e^{2\lambda}\right) \\
&= \lambda' e^{2\lambda} \\
\Gamma_{1,12} = \Gamma_{1,21} &= \frac{1}{2}\left(\partial_2 g_{11} + \partial_1 g_{21} - \partial_1 g_{12}\right) \\
&= 0 \\
\Gamma_{1,13} = \Gamma_{1,31} &= \frac{1}{2}\left(\partial_3 g_{11} + \partial_1 g_{31} - \partial_1 g_{13}\right) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_{1,22} &= \frac{1}{2}\left(\partial_2 g_{12} + \partial_2 g_{21} - \partial_1 g_{22}\right) \\
&= \frac{1}{2}\left(0 + 0 - (2r)\right) \\
&= -r \\
\Gamma_{1,23} = \Gamma_{1,32} &= \frac{1}{2}\left(\partial_3 g_{12} + \partial_2 g_{31} - \partial_1 g_{23}\right) \\
&= 0 \\
\Gamma_{1,33} &= \frac{1}{2}\left(\partial_3 g_{13} + \partial_3 g_{31} - \partial_1 g_{33}\right) \\
&= \frac{1}{2}\big(0 + 0 - \big(2r\sin^2\theta\big)\big) \\
&= -r\sin^2\theta \\
\Gamma_{2,00} &= \frac{1}{2}\left(\partial_0 g_{20} + \partial_0 g_{02} - \partial_2 g_{00}\right) \\
&= \frac{1}{2}\left(0 + 0 - 0\right) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_{2,01} = \Gamma_{2,10} &= \frac{1}{2}\left(\partial_1 g_{20} + \partial_0 g_{12} - \partial_2 g_{01}\right) \\
&= 0 \\
\Gamma_{2,02} = \Gamma_{2,20} &= \frac{1}{2}\left(\partial_2 g_{20} + \partial_0 g_{22} - \partial_2 g_{02}\right) \\
&= \frac{1}{2}\left(0 + 0 - 0\right) \\
&= 0 \\
\Gamma_{2,12} = \Gamma_{2,21} &= \frac{1}{2}\left(\partial_2 g_{21} + \partial_1 g_{22} - \partial_2 g_{12}\right) \\
&= \frac{1}{2}\left(0 + (2r) - 0\right) \\
&= r \\
\Gamma_{2,13} = \Gamma_{2,31} &= \frac{1}{2}\left(\partial_3 g_{21} + \partial_1 g_{32} - \partial_2 g_{13}\right) \\
&= 0
\end{aligned}
$$

$$\begin{aligned}
\Gamma_{2,22} = \Gamma_{2,21} &= \frac{1}{2}\left(\partial_2 g_{22} + \partial_2 g_{22} - \partial_2 g_{22}\right) \\
&= \frac{1}{2}(0) \\
&= 0 \\
\Gamma_{2,23} = \Gamma_{2,32} &= \frac{1}{2}\left(\partial_3 g_{22} + \partial_2 g_{32} - \partial_2 g_{23}\right) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\Gamma_{2,33} &= \frac{1}{2}\left(\partial_3 g_{23} + \partial_3 g_{32} - \partial_2 g_{33}\right) \\
&= \frac{1}{2}\Big(0 + 0 - \big(r^2.\sin\theta.\cos\theta\big)\Big) \\
&= -r^2.\sin\theta.\cos\theta \\
\Gamma_{3,00} &= \frac{1}{2}\left(\partial_0 g_{30} + \partial_0 g_{03} - \partial_3 g_{00}\right) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\Gamma_{3,01} = \Gamma_{3,10} &= \frac{1}{2}\left(\partial_1 g_{30} + \partial_0 g_{13} - \partial_3 g_{01}\right) \\
&= 0 \\
\Gamma_{3,02} = \Gamma_{3,20} &= \frac{1}{2}\left(\partial_2 g_{30} + \partial_0 g_{23} - \partial_3 g_{02}\right) \\
&= 0 \\
\Gamma_{3,03} = \Gamma_{3,30} &= \frac{1}{2}\left(\partial_3 g_{30} + \partial_0 g_{33} - \partial_3 g_{03}\right) \\
&= 0 \\
\Gamma_{3,11} &= \frac{1}{2}\left(\partial_1 g_{31} + \partial_1 g_{13} - \partial_3 g_{11}\right) \\
&= 0 \\
\Gamma_{3,12} = \Gamma_{3,21} &= \frac{1}{2}\left(\partial_2 g_{31} + \partial_1 g_{23} - \partial_3 g_{12}\right) \\
&= 0 \\
\Gamma_{3,13} = \Gamma_{3,31} &= \frac{1}{2}\left(\partial_3 g_{31} + \partial_1 g_{33} - \partial_3 g_{13}\right) \\
&= \frac{1}{2}\Big(0 + \big(2r\sin^2\theta\big) - 0\Big) \\
&= r\sin^2\theta
\end{aligned}$$

$$
\begin{aligned}
\Gamma_{3,22} &= \frac{1}{2}\left(\partial_2 g_{32} + \partial_2 g_{23} - \partial_3 g_{22}\right) \\
&= 0 \\
\Gamma_{3,23} = \Gamma_{3,32} &= \frac{1}{2}\left(\partial_3 g_{32} + \partial_2 g_{33} - \partial_3 g_{23}\right) \\
&= \frac{1}{2}\Big(0 + \big(r^2.\sin\theta\cos\theta\big) - 0\Big) \\
&= r^2 \sin\theta\cos\theta \\
\Gamma_{3,33} &= \frac{1}{2}\left(\partial_3 g_{33} + \partial_3 g_{33} - \partial_3 g_{33}\right) \\
&= 0
\end{aligned}
$$

Nilai simbol Christoffel Jenis ke-2 dapat dicari dengan menggunakan persamaan berikut:

$$
\Gamma^{\mu}_{\upsilon\rho} = g^{\mu\tau}\Gamma_{\tau,\upsilon\rho} \tag{A.80}
$$

Nilai-nilai tersebut adalah

$$
\begin{aligned}
\Gamma^{0}_{00} &= g^{00}\Gamma_{0,00} \\
&= 0 \\
\Gamma^{0}_{01} = \Gamma^{0}_{10} &= g^{0\tau}\Gamma_{\tau,01} \\
&= g^{00}\Gamma_{0,01} = -e^{(-2\nu)}.\left(-\nu' e^{(2\nu)}\right) = \nu' \\
&= g^{01}\Gamma_{1,01} = 0 \\
&= g^{02}\Gamma_{2,01} = 0 \\
&= g^{03}\Gamma_{3,01} = 0 \\
\Gamma^{0}_{02} = \Gamma^{0}_{20} &= g^{0\tau}\Gamma_{\tau,20} \\
&= g^{00}\Gamma_{0,20} \\
&= 0 \\
\Gamma^{0}_{03} = \Gamma^{0}_{30} &= g^{0\tau}\Gamma_{\tau,30} \\
&= g^{00}\Gamma_{0,30} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^0_{11} &= g^{0\tau}\Gamma_{\tau,11} \\
&= g^{00}\Gamma_{0,11} \\
&= 0 \\
\Gamma^0_{12} = \Gamma^0_{21} &= g^{0\tau}\Gamma_{\tau,12} \\
&= g^{00}\Gamma_{0,12} \\
&= 0 \\
\Gamma^0_{13} = \Gamma^0_{31} &= g^{0\tau}\Gamma_{\tau,13} \\
&= g^{00}\Gamma_{0,13} \\
&= 0 \\
\Gamma^0_{22} &= g^{0\tau}\Gamma_{\tau,22} \\
&= g^{00}\Gamma_{0,22} \\
&= 0 \\
\Gamma^0_{23} = \Gamma^0_{32} &= g^{0\tau}\Gamma_{\tau,23} \\
&= g^{00}\Gamma_{0,23} \\
&= 0 \\
\Gamma^0_{33} &= g^{0\tau}\Gamma_{\tau,33} \\
&= g^{00}\Gamma_{0,33} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^1_{00} &= g^{1\tau}\Gamma_{\tau,00} \\
&= g^{11}\Gamma_{1,00} \\
&= e^{(-2\lambda)}.\left(\nu' e^{(2\nu)}\right) \\
&= \nu' e^{(2\nu-2\lambda)} \\
\Gamma^1_{01} = \Gamma^1_{10} &= g^{1\tau}\Gamma_{\tau,01} \\
&= g^{11}\Gamma_{1,01} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^1_{02} = \Gamma^1_{20} &= g^{1\tau}\Gamma_{\tau,02} \\
&= g^{11}\Gamma_{1,02} \\
&= 0 \\
\Gamma^1_{03} = \Gamma^1_{30} &= g^{1\tau}\Gamma_{\tau,03} \\
&= g^{11}\Gamma_{1,03} \\
&= 0 \\
\Gamma^1_{11} &= g^{1\tau}\Gamma_{\tau,11} \\
&= g^{11}\Gamma_{1,11} \\
&= e^{(-2\lambda)}.\Big(\lambda' e^{(2\lambda)}\Big) \\
&= \lambda' \\
\Gamma^1_{12} = \Gamma^1_{21} &= g^{1\tau}\Gamma_{\tau,12} \\
&= g^{11}\Gamma_{1,12} \\
&= 0 \\
\Gamma^1_{13} = \Gamma^1_{31} &= g^{1\tau}\Gamma_{\tau,13} \\
&= g^{11}\Gamma_{1,13} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^1_{22} &= g^{1\tau}\Gamma_{\tau,22} \\
&= g^{11}\Gamma_{1,22} \\
&= e^{(-2\lambda)}.\,(-r) \\
&= -re^{(-2\lambda)} \\
\Gamma^1_{23} = \Gamma^1_{32} &= g^{1\tau}\Gamma_{\tau,23} \\
&= g^{11}\Gamma_{1,23} \\
&= 0 \\
\Gamma^1_{33} &= g^{1\tau}\Gamma_{\tau,33} \\
&= g^{11}\Gamma_{1,33} \\
&= e^{(-2\lambda)}.\,\left(-r\sin^2\theta\right) \\
&= -r\sin^2\theta e^{(-2\lambda)}
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^2_{00} &= g^{1\tau}\Gamma_{\tau,00} \\
&= g^{22}\Gamma_{2,00} \\
&= 0 \\
\Gamma^2_{01} = \Gamma^2_{10} &= g^{2\tau}\Gamma_{\tau,01} \\
&= g^{22}\Gamma_{2,01} \\
&= 0 \\
\Gamma^2_{02} = \Gamma^2_{20} &= g^{2\tau}\Gamma_{\tau,02} \\
&= g^{22}\Gamma_{2,02} \\
&= 0 \\
\Gamma^2_{03} = \Gamma^2_{30} &= g^{2\tau}\Gamma_{\tau,03} \\
&= g^{22}\Gamma_{2,03} \\
&= 0 \\
\Gamma^2_{11} &= g^{22}\Gamma_{2,11} \\
&= 0 \\
\Gamma^2_{12} = \Gamma^2_{21} &= g^{2\tau}\Gamma_{\tau,12} \\
&= g^{22}\Gamma_{2,12} \\
&= (r)^{-2}\,.\,(r) \\
&= \frac{1}{r} \\
\Gamma^2_{13} = \Gamma^2_{31} &= g^{22}\Gamma_{2,13} \\
&= 0 \\
\Gamma^2_{22} = \Gamma^2_{22} &= g^{22}\Gamma_{2,22} \\
&= 0 \\
\Gamma^2_{23} = \Gamma^2_{32} &= g^{22}\Gamma_{2,23} \\
&= 0 \\
\Gamma^2_{33} = \Gamma^2_{33} &= g^{22}\Gamma_{2,33} \\
&= (r)^{-2}\,.\,\left(-r^2\sin\theta\cos\theta\right) \\
&= -\sin\theta\cos\theta
\end{aligned}
$$

$$\begin{aligned}
\Gamma^3_{00} &= g^{33}\Gamma_{3,00} \\
&= 0 \\
\Gamma^3_{01} = \Gamma^3_{10} &= g^{33}\Gamma_{3,01} \\
&= 0 \\
\Gamma^3_{02} = \Gamma^3_{20} &= g^{33}\Gamma_{3,02} \\
&= 0 \\
\Gamma^3_{03} = \Gamma^3_{10} &= g^{33}\Gamma_{3,03} \\
&= 0 \\
\Gamma^3_{11} &= g^{33}\Gamma_{3,11} \\
&= 0 \\
\Gamma^3_{12} = \Gamma^3_{21} &= g^{33}\Gamma_{3,12} \\
&= 0 \\
\Gamma^3_{13} = \Gamma^3_{31} &= g^{33}\Gamma_{3,13} \\
&= r^{-2}\sin^{-2}\theta.\left(r\sin^2\theta\right) \\
&= \frac{1}{r} \\
\Gamma^3_{23} = \Gamma^3_{32} &= g^{33}\Gamma_{3,23} \\
&= r^{-2}\sin^{-2}\theta.\left(r^2\sin\theta.\cos\theta\right) \\
&= \frac{\sin\theta}{\cos\theta} \\
&= \cot\theta \\
\Gamma^3_{33} &= g^{33}\Gamma_{3,33} \\
&= 0
\end{aligned}$$

Dari 64 komponen komponen simbol Christoffel di atas, komponen-komponen yang tidak bernilai 0 adalah:

$$\begin{aligned}
\Gamma^0_{01} &= \Gamma^0_{10} = \nu' \\
\Gamma^1_{00} &= \nu' e^{(2\nu - 2\lambda)} \\
\Gamma^1_{11} &= \lambda' \\
\Gamma^1_{22} &= -re^{-2\lambda} \\
\Gamma^1_{33} &= -r\sin^2\theta\, e^{-2\lambda} \\
\Gamma^2_{12} &= \Gamma^2_{21} = \frac{1}{r} \\
\Gamma^2_{33} &= -\sin\theta\cos\theta \\
\Gamma^3_{13} &= \Gamma^3_{31} = \frac{1}{r} \\
\Gamma^3_{23} &= \Gamma^3_{32} = \cot\theta
\end{aligned} \tag{A.81}$$

Tensor Ricci dapat dicari dengan menggunakan pers.(A.27)

$$R_{\tau\upsilon} = \partial\upsilon\Gamma^\gamma_{\tau\gamma} - \partial\gamma\Gamma^\gamma_{\tau\upsilon} + \Gamma^\rho_{\tau\gamma}\Gamma^\gamma_{\rho\upsilon} - \Gamma^\rho_{\tau\upsilon}\Gamma^\gamma_{\rho\gamma} \tag{A.82}$$

Tensor Ricci bersifat simetri ($R_{\mu\nu} = R_{\nu\mu}$), maka hanya memiliki 10 komponen bebas. Untuk komponen $R_{i0}, (i = 1, 2, 3)$ :

$$R_{i0} = \partial_0\Gamma^\sigma_{i\sigma} - \partial_\sigma\Gamma^\sigma_{i0} + \Gamma^\rho_{i\sigma}\Gamma^\sigma_{\rho 0} - \Gamma^\rho_{i0}\Gamma^\sigma_{\rho\sigma} \tag{A.83}$$

dengan kondisi statik mensyaratkan bahwa $\partial_0 g_{\mu\nu} \equiv 0$ sehingga $\partial_0\Gamma^\sigma_{i\sigma} \equiv 0$. Tensor Ricci menjadi

$$R_{i0} = -\partial_j\Gamma^j_{i0} + \Gamma^\rho_{ij}\Gamma^j_{\rho 0} - \Gamma^\rho_{i0}\Gamma^j_{\rho j} \tag{A.84}$$

Dengan menggunakan nilai $\Gamma^i_{j0} = 0$, $\Gamma^\rho_{0\rho} = 0$ dan $\Gamma^0_{ij} = 0$, maka didapatkan

$$R_{i0} = R_{0i} = 0 \tag{A.85}$$

sehingga komponen tensor Ricci yang tersisa adalah komponen dalam arah diagonal ($R_{\tau\tau}$). Nilai dari $R_{\tau\tau}$ ini adalah

$$R_{\tau\tau} = \partial_\tau \Gamma^\gamma_{\tau\gamma} - \partial\gamma \Gamma^\gamma_{\tau\tau} + \Gamma^\rho_{\tau\gamma}\Gamma^\gamma_{\rho\tau} - \Gamma^\rho_{\tau\tau}\Gamma^\gamma_{\rho\gamma} \qquad \text{(A.86)}$$

untuk $\tau = 0$

$$\begin{aligned}
R_{00} &= \partial_0 \Gamma^\gamma_{0\gamma} - \partial\gamma \Gamma^\gamma_{00} + \Gamma^\rho_{0\gamma}\Gamma^\gamma_{\rho 0} - \Gamma^\rho_{00}\Gamma^\gamma_{\rho\gamma} \\
&= 0 - \partial_1 \Gamma^1_{00} + \left(\Gamma^0_{0\gamma}\Gamma^\gamma_{00} + \Gamma^1_{0\gamma}\Gamma^\gamma_{01}\right) - \Gamma^1_{00}\Gamma^\gamma_{1\gamma} \\
&= -\partial_1 \Gamma^1_{00} + \left(\Gamma^0_{0\gamma}\Gamma^\gamma_{00} + \Gamma^1_{0\gamma}\Gamma^\gamma_{01}\right) - \Gamma^1_{00}\left(\nu' + \lambda' + \frac{2}{r}\right) \\
&= -\partial_r \left(\nu' e^{(2\nu - 2\lambda)} + 2\nu'^2 e^{(2\nu-2\lambda)}\right) - \nu' e^{(2\nu-2\lambda)}\left(\nu' + \lambda' + \frac{1}{r}\right) \\
&= -\nu'' e^{(2\nu-2\lambda)} - \nu'\left(2\nu' - 2\lambda'\right) e^{(2\nu-2\lambda)} + 2\nu' e^{(2\nu-2\lambda)} \\
&\quad - \left(\nu' + \lambda' + \frac{2}{r}\right)\nu' e^{(2\nu-2\lambda)} \\
&= \left\{-\nu'' - \nu'.2\nu' + \nu'.2\lambda' + 2\nu'^2 - \nu'^2 + \nu'\lambda' + \frac{2\nu'}{r}\right\} e^{(2\nu-2\lambda)} \\
&= \left\{-\nu'' + \nu'\lambda' - \nu'^2 - \frac{2\nu'}{r}\right\} e^{(2\nu-2\lambda)} \qquad \text{(A.87)}
\end{aligned}$$

untuk $\tau = 1$

$$\begin{aligned}
R_{11} &= \partial_1\Gamma^{\gamma}_{1\gamma} - \partial_\gamma\Gamma^{\gamma}_{11} + \Gamma^{\rho}_{1\gamma}\Gamma^{\gamma}_{\rho 1} - \Gamma^{\rho}_{11}\Gamma^{\gamma}_{\rho\gamma} \\
&= \partial_1\left(\Gamma^0_{10} + \Gamma^1_{11} + \Gamma^2_{21} + \Gamma^3_{13}\right) - \left(\partial_0\Gamma^0_{11}\partial_1\Gamma^1_{11} + \partial_2\Gamma^2_{11} + \partial_3\Gamma^3_{11}\right) \\
&\quad + \left\{\Gamma^0_{1\gamma}\Gamma^{\gamma}_{01} + \Gamma^1_{1\gamma}\Gamma^{\gamma}_{11} + \Gamma^2_{1\gamma} + \Gamma^{\gamma}_{21} + \Gamma^3_{1\gamma}\Gamma^{\gamma}_{31}\right\} \\
&\quad - \Gamma^1_{11}\left(\Gamma^1_{11} + \Gamma^2_{12} + \Gamma^3_{13}\right) - \Gamma^2_{11}\left(\Gamma^1_{21} + \Gamma^2_{22} + \Gamma^3_{23}\right) \\
&\quad - \Gamma^3_{11}\left(\Gamma^1_{31} + \Gamma^2_{32} + \Gamma^3_{33}\right) - \Gamma^0_{11}\left(\Gamma^1_{01} + \Gamma^2_{02} + \Gamma^3_{03}\right) \\
&= \partial_1\left(\Gamma^0_{10} + \Gamma^1_{11} + \Gamma^2_{21} + \Gamma^3_{13}\right) - \left(\partial_0\Gamma^0_{11}\partial_1\Gamma^1_{11} + \partial_2\Gamma^2_{11} + \partial_3\Gamma^3_{11}\right) \\
&\quad + \left(\Gamma^0_{11}\Gamma^1_{01} + \Gamma^0_{12}\Gamma^2_{01} + \Gamma^0_{13}\Gamma^3_{01} + \Gamma^0_{10}\Gamma^0_{01}\right) \\
&\quad + \left(\Gamma^1_{10}\Gamma^0_{11} + \Gamma^1_{11}\Gamma^1_{11} + \Gamma^1_{12}\Gamma^2_{11} + \Gamma^1_{13}\Gamma^3_{11}\right) \\
&\quad + \left(\Gamma^2_{10}\Gamma^0_{21} + \Gamma^2_{11}\Gamma^1_{21} + \Gamma^2_{12}\Gamma^2_{21} + \Gamma^3_{13}\Gamma^3_{21}\right) \\
&\quad + \left(\Gamma^3_{10}\Gamma^0_{31} + \Gamma^3_{11}\Gamma^1_{31} + \Gamma^3_{12}\Gamma^2_{31} + \Gamma^3_{13}\Gamma^3_{31}\right) \\
&\quad - \Gamma^1_{11}\left(\Gamma^1_{11} + \Gamma^2_{12} + \Gamma^3_{13} + \Gamma^0_{10}\right) - \Gamma^2_{11}\left(\Gamma^1_{21} + \Gamma^2_{22} + \Gamma^3_{23}\right) \\
&\quad - \Gamma^3_{11}\left(\Gamma^1_{31} + \Gamma^2_{32} + \Gamma^3_{33}\right) - \Gamma^0_{11}\left(\Gamma^1_{01} + \Gamma^2_{02} + \Gamma^3_{03}\right)
\end{aligned}$$

$$\begin{aligned}
R_{11} &= \partial_1\left(\Gamma^0_{10} + \Gamma^1_{11} + \Gamma^2_{21} + \Gamma^3_{13}\right) - \partial_1\Gamma^1_{11} \\
&\quad + \left(\Gamma^0_{10}\Gamma^0_{10} + \Gamma^1_{11}\Gamma^1_{11} + \Gamma^2_{12}\Gamma^2_{12} + \Gamma^3_{13}\right) - \Gamma^1_{11}\left(\Gamma^1_{11} + \Gamma^2_{12} + \Gamma^3_{13} + \Gamma^0_{0}\right) \\
&= \left(\nu'' + \lambda'' + \left(-\frac{1}{r^2}\right) + \left(-\frac{1}{r^2}\right)\right) - \lambda'' \\
&\quad + \left[(\nu')^2 + (\lambda')^2 + \left(\frac{1}{r}\right)^2 + \left(\frac{1}{r}\right)^2\right] - \lambda'\left(\nu' + \lambda' + \frac{1}{r} + \frac{1}{r}\right) \\
&= \nu'' + \lambda'' - \frac{2}{r^2} - \lambda'' + (\nu')^2 + (\lambda')^2 + \frac{2}{r^2} - \lambda'\left(\nu' + \lambda' + \frac{2}{r}\right) \\
R_{11} &= \nu'' + (\nu')^2 - \lambda'\nu' - \frac{2\lambda'}{r} \qquad \text{(A.88)}
\end{aligned}$$

untuk $\tau = 2$

$$
\begin{aligned}
R_{22} &= \partial_2\Gamma^{\gamma}_{2\gamma} - \partial_\gamma\Gamma^{\gamma}_{22} + \Gamma^{\rho}_{2\gamma}\Gamma^{\gamma}_{\rho 2} - \Gamma^{\rho}_{22}\Gamma^{\gamma}_{\rho\gamma} \\
&= \partial_2\left(\Gamma^0_{20} + \Gamma^1_{21} + \Gamma^2_{22} + \Gamma^3_{23}\right) - \left(\partial_1\Gamma^1_{22} + \partial_2\Gamma^2_{22} + \partial_3\Gamma^3_{22} + \partial_0\Gamma^0_{22}\right) \\
&\quad + \left\{\Gamma^0_{2\gamma}\Gamma^{\gamma}_{02} + \Gamma^1_{2\gamma}\Gamma^{\gamma}_{12} + \Gamma^2_{2\gamma}\Gamma^{\gamma}_{22} + \Gamma^3_{2\gamma}\Gamma^{\gamma}_{32}\right\} \\
&\quad - \left\{\Gamma^0_{22}\Gamma^{\gamma}_{0\gamma} + \Gamma^1_{22}\Gamma^{\gamma}_{1\gamma} + \Gamma^2_{22}\Gamma^{\gamma}_{2\gamma} + \Gamma^3_{22}\Gamma^{\gamma}_{3\gamma}\right\} \\
&= \partial_2\left(\Gamma^0_{20} + \Gamma^1_{21} + \Gamma^2_{22} + \Gamma^3_{23}\right) - \left(\partial_1\Gamma^1_{22} + \partial_2\Gamma^2_{22} + \partial_3\Gamma^3_{22} + \partial_0\Gamma^0_{22}\right) \\
&\quad + \left(\Gamma^0_{20}\Gamma^0_{02} + \Gamma^0_{21}\Gamma^1_{02} + \Gamma^0_{22}\Gamma^2_{02} + \Gamma^0_{23}\Gamma^3_{02}\right) \\
&\quad + \left(\Gamma^1_{20}\Gamma^{0\gamma}_{12} + \Gamma^1_{21}\Gamma^1_{12} + \Gamma^1_{22}\Gamma^2_{12} + \Gamma^1_{23}\Gamma^3_{12}\right) \\
&\quad + \left(\Gamma^2_{20}\Gamma^0_{22} + \Gamma^2_{21}\Gamma^1_{22} + \Gamma^2_{22}\Gamma^2_{22} + \Gamma^2_{23}\Gamma^3_{22}\right) \\
&\quad + \left(\Gamma^3_{20}\Gamma^0_{23} + \Gamma^3_{21}\Gamma^1_{23} + \Gamma^3_{22}\Gamma^2_{23} + \Gamma^3_{23}\Gamma^3_{23}\right) \\
&\quad - \left[\Gamma^1_{22}\left(\Gamma^0_{10} + \Gamma^1_{11} + \Gamma^2_{12} + \Gamma^3_{13}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
R_{22} &= -\csc^2\theta + e^{(-2\lambda)} - 2r\lambda' e^{(-2\lambda)} - 2e^{(-2\lambda)} \\
&\quad + re^{(-2\lambda)}\left(\nu' + \lambda' + \frac{2}{r}\right) + \cot^2\theta \\
&= \left(1 - 2r\lambda' - 2 + r\nu' + r\lambda' + 2\right)e^{(-2\lambda)} - \csc^2\theta + \cot^2\theta \\
&= \left(1 + r\nu' - r\lambda'\right)e^{(-2\lambda)} - 1 \qquad \text{(A.89)}
\end{aligned}
$$

untuk $\tau = 3$

$$
\begin{aligned}
R_{33} &= \partial_3\Gamma^{\gamma}_{3\gamma} - \partial_\gamma\Gamma^{\gamma}_{33} + \Gamma^{\rho}_{3\gamma}\Gamma^{\gamma}_{\rho 3} - \Gamma^{\rho}_{33}\Gamma^{\gamma}_{\rho\gamma} \\
&= \partial_3\left(\Gamma^0_{30} + \Gamma^1_{31} + \Gamma^2_{32} + \Gamma^3_{33}\right) - \left(\partial_0\Gamma^0_{33} + \partial_1\Gamma^1_{33} + \partial_2\Gamma^2_{33} + \partial_3\Gamma^3_{33}\right) \\
&\quad + \left\{\Gamma^0_{3\gamma}\Gamma^{\gamma}_{03} + \Gamma^1_{3\gamma}\Gamma^{\gamma}_{13} + \Gamma^2_{3\gamma}\Gamma^{\gamma}_{23} + \Gamma^3_{3\gamma}\Gamma^{\gamma}_{33}\right\} \\
&\quad - \left\{\Gamma^0_{33}\Gamma^{\gamma}_{0\gamma} + \Gamma^1_{33}\Gamma^{\gamma}_{1\gamma} + \Gamma^2_{33}\Gamma^{\gamma}_{2\gamma} + \Gamma^3_{33}\Gamma^{\gamma}_{3\gamma}\right\} \\
&= -\partial_1\Gamma^1_{33} - \partial_2\Gamma^2_{33} + \left(\Gamma^0_{30}\Gamma^0_{03} + \Gamma^0_{31}\Gamma^1_{03} + \Gamma^0_{32}\Gamma^2_{03} + \Gamma^0_{33}\Gamma^3_{03}\right) \\
&\quad + \left(\Gamma^1_{30}\Gamma^0_{13} + \Gamma^1_{31}\Gamma^1_{13} + \Gamma^1_{32}\Gamma^2_{13} + \Gamma^1_{33}\Gamma^3_{13}\right) \\
&\quad + \left(\Gamma^2_{30}\Gamma^0_{23} + \Gamma^2_{31}\Gamma^1_{23} + \Gamma^2_{32}\Gamma^2_{23} + \Gamma^2_{33}\Gamma^3_{23}\right) \\
&\quad + \left(\Gamma^3_{30}\Gamma^0_{33} + \Gamma^3_{31}\Gamma^1_{33} + \Gamma^3_{32}\Gamma^2_{33} + \Gamma^3_{33}\Gamma^3_{33}\right) \\
&\quad -\Gamma^1_{33}\left(\Gamma^0_{10} + \Gamma^1_{11} + \Gamma^2_{12} + \Gamma^3_{13}\right) \\
&\quad -\Gamma^2_{33}\left(\Gamma^0_{20} + \Gamma^1_{21} + \Gamma^2_{22} + \Gamma^3_{23}\right) \\
&= \sin^2\theta e^{(-2\lambda)} - 2r\lambda'\sin^2\theta e^{(-2\lambda)} - \left(-\cos^2\theta + \sin^2\theta\right) \\
&\quad +2\left(-r\sin^2\theta e^{(-2\lambda)}\frac{1}{r}\right) + 2\left(-\sin\theta\cos\theta.\frac{\cos\theta}{\sin\theta}\right) \\
&\quad - \left(-r\sin^2\theta e^{(-2\lambda)}\left(\nu' + \lambda' + \frac{2}{r}\right)\right) - \left\{-\sin\theta\cos\theta.\frac{\cos\theta}{\sin\theta}\right\} \\
&= \sin^2\theta\left(1 - 2r\lambda' - 2 + r\nu' + r\lambda' + 2\right)e^{(-2\lambda)} \\
&\quad + \left(\cos^2\theta - \sin^2\theta - 2\cos^2\theta + \cos^2\theta\right)
\end{aligned}
$$

$$
\begin{aligned}
R_{33} &= \sin^2\theta\left[\left(1 + r\nu' - r\lambda'\right)e^{(-2\lambda)} - 1\right] \\
&= \sin^2\theta R_{22} \qquad \text{(A.90)}
\end{aligned}
$$

Untuk kondisi dimana tidak ada materi dan energi (vakum), ($R_{\mu\nu} = 0$), maka pers.(A.87), pers.(A.88), pers.(A.89) menjadi

$$-\nu'' + \nu'\lambda' - \nu'^2 - \frac{2}{r}\nu' = 0 \tag{A.91}$$

$$\nu'' - \nu'\lambda' + \nu'^2 - \frac{2}{r}\lambda' = 0 \tag{A.92}$$

$$(1 + r\nu' - r\lambda')\, e^{(-2\lambda)} = 1 \tag{A.93}$$

dengan menjumlahkan pers.(3.18) dan pers.(3.19)

$$\frac{-2}{r}(\nu' + \lambda') = 0$$

atau

$$\begin{aligned} (\nu' + \lambda') &= 0 \\ \nu + \lambda &= \text{konstan} \end{aligned} \tag{A.94}$$

Pada $r \to \infty$, metrik harus kembali pada bentuk Minkowski $\eta_{\mu\nu}$ sehingga $\nu$ dan $\lambda \to 0$.
maka

$$\nu + \lambda = 0 \to \nu = -\lambda \tag{A.95}$$

dengan memasukkan pers.(3.22) ke pers.(3.20)

$$(1 + 2r\nu')\, e^{(2\nu)} = \frac{d}{dr}\left[re^{(2\nu)}\right] = 1 \tag{A.96}$$

dengan mengintegralkan pers.(3.23), maka

$$\int d\left[re^{(2\nu)}\right] = \int dr\; re^{(2\nu)} = r + C \tag{A.97}$$

dengan $C$ merupakan konstanta integrasi. Konstanta ini dihitung dengan pendekatan untuk medan lemah. Lagrangian non-relativistik berbentuk:

$$
\begin{aligned}
L &= -mc^2 - m\phi + \frac{1}{2}mv^2 \\
&= -mc\left( c + \frac{\phi}{c} - \frac{1}{2}\frac{v^2}{c}\right) \\
&= -mc\left( c + \frac{\phi}{c} - \frac{1}{2}g_{ij}\frac{v^i}{c}v^j\right)
\end{aligned} \tag{A.98}
$$

Pada koordinat kartesian, $g_{ij} = -\delta_{ij}$ sehingga pers.(A.98) menjadi:

$$
L = -mc\left( c + \frac{\phi}{c} + \frac{1}{2}g_{ij}\frac{v^i}{c}v^j\right) \tag{A.99}
$$

Integral aksi adalah

$$
\begin{aligned}
I &= \int L dt \\
&= -mc\int\left( c + \frac{\phi}{c} + \frac{1}{2}g_{ij}\frac{v^i}{c}v^j\right) dt \\
&= -mc\int\left( \left(c + \frac{\phi}{c}\right) + \frac{1}{2}g_{ij}\frac{v^i}{c}\frac{dx^j}{dt}\right) dt \\
&= -mc\int\left( \left(1 + \frac{\phi}{c^2}\right) cdt + \frac{1}{2}g_{ij}\frac{v^i}{c}dx^j\right) \\
&= -mc\int ds
\end{aligned} \tag{A.100}
$$

maka

$$
ds = \left(1 + \frac{\phi}{c^2}\right) cdt + \frac{1}{2}g_{ij}\frac{v^i}{c}dx^j \tag{A.101}
$$

sehingga

$$\begin{aligned} ds^2 &= \left(1+\frac{\phi}{c^2}\right)^2 c^2dt^2 + \left(1+\frac{\phi}{c^2}\right) g_{ij}\frac{v^i}{c}dx^j cdt + \underbrace{\frac{1}{4}\left(g_{ij}\frac{v^i}{c}dx^j\right)^2}_{\approx 0} \\ &\approx \left(1+\frac{\phi}{c^2}\right)^2 c^2dt^2 + \left(1+\frac{\phi}{c^2}\right) g_{ij}dx^idx^j \end{aligned} \tag{A.102}$$

dengan:

$$\left(1+\frac{\phi}{c^2}\right)^2 = 1+\frac{2\phi}{c^2}+\frac{\phi^2}{c^4} \approx 1+\frac{2\phi}{c^2} \tag{A.103}$$

serta

$$\left(1+\frac{\phi}{c^2}\right) g_{ij}dx^idx^j \approx g_{ij}dx^idx^j \tag{A.104}$$

sehingga

$$ds^2 \approx \underbrace{\left(1+\frac{2\phi}{c^2}\right)}_{g_{00}} c^2dt^2 + g_{ij}dx^idx^j \tag{A.105}$$

dibandingkan solusi Schawrszchild

$$g_{00} = 1+\frac{C}{r} \tag{A.106}$$

maka

$$\begin{aligned}
1+\frac{C}{r} &= 1+\frac{2\phi}{c^2} \\
C &= \phi\frac{2r}{c^2} \\
C &= -\frac{GM}{r}\frac{2r}{c^2} \\
C &= -\frac{2GM}{c^2} \\
C &\equiv -2m
\end{aligned} \tag{A.107}$$

dengan $m \equiv \frac{GM}{c^2}$. Maka elemen garisnya akan menjadi:

$$ds^2 = -\left(1-\frac{2m}{r}\right)c^2\left(dt\right)^2+\left(1-\frac{2m}{r}\right)^{-1}\left(dr\right)^2+r^2d\Omega^2 \tag{A.108}$$

Persamaan tersebut merupakan metrik Schwarzschild.

## A.6 Solusi Reissner-Nordstrom

Sama seperti pada kasus Schwarzschild, untuk benda bermassa $M$ dan bermuatan $Q$. Persamaan medan Einstein (A.68)

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T\right) \tag{A.109}$$

dengan tensor energi-momentumnya adalah

$$T_{\mu\nu} = -F_{\mu\rho}F_{\nu}^{\rho}+\frac{1}{4}g_{\mu\nu}F_{\rho\lambda}F^{\rho\lambda} \tag{A.110}$$

Nilai $T$ pada pers.(A.109) adalah

$$\begin{aligned}
T &= g^{\mu\nu}T_{\mu\nu} \\
&= -g^{\mu\nu}F_{\nu}{}^{\rho}F_{\mu\rho} + \frac{1}{4}g^{\mu\nu}g_{\mu\nu}F_{\rho\lambda}F^{\rho\lambda} \\
&= -F^{\mu\rho}F_{\mu\rho} + \frac{1}{4}\delta^{\mu}_{\mu}F_{\rho\lambda}F^{\rho\lambda} \\
&= -F^{\mu\rho}F_{\mu\rho} + \frac{1}{4}\left(\delta^0_0 + \delta^1_1 + \delta^2_2 + \delta^3_3\right)F_{\rho\lambda}F^{\rho\lambda} \\
&= -F^{\mu\rho}F_{\mu\rho} + \frac{1}{4}.4F_{\rho\lambda}F^{\rho\lambda} \\
&= -F^{\mu\rho}F_{\mu\rho} + \underbrace{F_{\rho\lambda}F^{\rho\lambda}}_{\rho\to\mu,\lambda\to\nu} \\
&= -F^{\mu\rho}F_{\mu\rho} + F_{\mu\rho}F^{\mu\rho} \\
&= 0
\end{aligned} \tag{A.111}$$

sehingga perssamaan medan Einstein (A.109) menjadi

$$R_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{A.112}$$

Tensor energi-momentum dapat dicari jika tensor kuat medan diketahui. Tensor kuat medan $F_{\mu\nu}$ bersifat anti-simetri dengan

$$\partial_{[\rho}F_{\mu\nu]} = 0 \tag{A.113}$$

Sehingga

$$\begin{aligned}
\frac{1}{3!}\left(\partial_\rho F_{\mu\nu} - \partial_\rho F_{\nu\mu} + \partial_\nu F_{\rho\mu} - \partial_\nu F_{\mu\rho} + \partial_\mu F_{\nu\rho} - \partial_\mu F_{\rho\nu}\right) &= 0 \\
\left(\partial_\rho F_{\mu\nu} + \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} + \partial_\nu F_{\rho\mu} + \partial_\mu F_{\nu\rho} + \partial_\mu F_{\nu\rho}\right) &= 0 \\
\left(2\partial_\rho F_{\mu\nu} + 2\partial_\nu F_{\rho\mu} + 2\partial_\mu F_{\nu\rho}\right) &= 0
\end{aligned} \tag{A.114}$$

maka

$$\partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} + \partial_\mu F_{\nu\rho} = 0 \tag{A.115}$$

untuk $\rho = 0$, maka pers.(A.115) menjadi

$$\begin{aligned}
\partial_0 F_{\mu\nu} + \underbrace{\partial_\nu F_{0\mu} + \partial_\mu F_\nu}_{\mu\leftrightarrow\nu} &= 0 \\
\partial_0 F_{\mu\nu} + \partial_\nu F_{0\mu} + \partial_\nu F_\mu &= 0 \\
\partial_0 F_{\mu\nu} + \partial_\nu F_{0\mu} - \partial_\nu F_{0\mu} &= 0 \\
\partial_0 F_{\mu\nu} &= 0 \qquad \text{(A.116)}
\end{aligned}$$

medan listrik di luar benda bermuatan total $q$ hanya dalam arah radial

$$E = E(r) = \frac{q}{4\pi\epsilon_0 r^2} \equiv f \qquad \text{(A.117)}$$

maka hanya komponen $F_{01}$ dan $F_{10}$ yang tidak bernilai nol yaitu

$$F_{01} = f(r) = -F_{10} \qquad \text{(A.118)}$$

Sehingga tensor kuat medan $F_{\mu\nu}$ adalah

$$F_{\mu\nu} = \begin{pmatrix} 0 & f & 0 & 0 \\ -f & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{(A.119)}$$

Bentuk kontravarian dari tensor kuat medan di atas adalah

$$F^{\rho\lambda} = g^{\rho\mu} g^{\lambda\nu} F_{\mu\nu} \qquad \text{(A.120)}$$

dengan komponen-komponen kontravarian yang tidak nol yaitu

$$\begin{aligned}
F^{01} &= g^{0\mu} g^{1\nu} F_{\mu\nu} \\
&= g^{00} g^{10} F_{01} \\
&= -e^{-2\nu}.e^{2\lambda}.f \\
&= -fe^{-2\nu-2\lambda}
\end{aligned} \qquad \text{(A.121)}$$

$$\begin{aligned} F^{10} &= g^{1\mu}g^{1\nu}F_{\mu\nu} \\ &= g^{11}g^{00}F_{10} \\ &= -e^{-2\lambda}.e^{2\nu}. - f \\ &= fe^{-2\nu-2\lambda} \\ &= -F^{01} \end{aligned} \tag{A.122}$$

sehingga tensor kuat medan dalam bentuk kontravarian adalah

$$F^{\mu\nu} = \begin{pmatrix} 0 & -fe^{-2\nu-2\lambda} & 0 & 0 \\ fe^{-2\nu-2\lambda} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{A.123}$$

Bentuk tensor campuran kuat medan adalah

$$F_{\nu}{}^{\rho} = g_{\nu\gamma}F^{\gamma\rho} \tag{A.124}$$

dengan komponen-komponen yang tidak bernilai nol

$$\begin{aligned} F_{0}{}^{1} &= g_{0\gamma}F^{\gamma 1} \\ &= g_{00}F^{01} \\ &= -e^{2\nu}(-fe^{-2\nu-2\lambda}) \\ &= fe^{-2\lambda} \end{aligned} \tag{A.125}$$

$$\begin{aligned} F_{1}{}^{0} &= g_{1\gamma}F^{\gamma 0} \\ &= g_{11}F^{10} \\ &= e^{2\lambda}(fe^{-2\nu-2\lambda}) \\ &= fe^{-2\nu} \end{aligned} \tag{A.126}$$

sehingga tensor kuat medan dalam bentuk tensor campuran adalah

$$F_{\nu}{}^{\rho} = \begin{pmatrix} 0 & fe^{-2\nu} & 0 & 0 \\ fe^{-2\lambda} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{A.127}$$

Dari pers.(A.110), nilai komponen diagonal dari tensor energi-momentum $T_{\mu\nu)}$ adalah

$$T_{\mu\nu} = -F_{\mu\rho}F_{\nu}{}^{\rho} + \frac{1}{4}g_{\mu\nu}F_{\rho\lambda}F^{\rho\lambda} \tag{A.128}$$

$$\begin{aligned} T_{00} &= -F_{0\rho}F_{0}{}^{\rho} + \frac{1}{4}g_{00}F_{\rho\lambda}F^{\rho\lambda} \\ &= -F_{01}F_{0}{}^{1} + \frac{1}{4}g_{00}(F_{0\sigma}F^{0\sigma} + F_{1\sigma}F^{1\sigma}) \\ &= -F_{01}F_{0}{}^{1} + \frac{1}{4}g_{00}(F_{01}F^{01} + F_{01}F^{01}) \\ &= -F_{01}F_{0}{}^{1} + \frac{1}{4}g_{00}2F_{01}F^{01} \\ &= -F_{01}F_{0}{}^{1} + \frac{1}{2}(-e^{2\nu})F_{01}F^{01} \\ &= -f.fe^{-2\lambda} + \frac{1}{2}(-e^{2\nu})f(-fe^{-2\nu-2\lambda}) \\ &= -f^{2}e^{-2\lambda} + \frac{1}{2}f^{2}(e^{-2\lambda}) \\ &= -\frac{1}{2}f^{2}(e^{-2\lambda}) \end{aligned} \tag{A.129}$$

$$
\begin{aligned}
T_{11} &= -F_{1\rho}F_1{}^{\rho} + \frac{1}{4}g_{11}F_{\rho\sigma}F^{\rho\sigma} \\
&= -F_{10}F_1{}^{0} + \frac{1}{4}g_{11}(F_{0\sigma}F^{0\sigma} + F_{1\sigma}F^{1\sigma}) \\
&= -(-f)fe^{-2\nu} + \frac{1}{4}g_{11}(F_{01}F^{01} + F_{10}F^{10}) \\
&= f^2e^{-2\nu} + \frac{1}{2}g_{11}F_{01}F^{01} \\
&= f^2e^{-2\nu} + \frac{1}{2}e^{-2\lambda}f(-fe^{-2\lambda-2\nu}) \\
&= f^2e^{-2\nu} - \frac{1}{2}f^2e^{-2\nu} \\
&= \frac{1}{2}f^2e^{-2\nu}
\end{aligned}
\tag{A.130}
$$

$$
\begin{aligned}
T_{22} &= -F_{2\rho}F_2{}^{\rho} + \frac{1}{4}g_{22}F_{\rho\sigma}F^{\rho\sigma} \\
&= \frac{1}{4}g_{22}F_{\rho\sigma}F^{\rho\sigma} \\
&= \frac{1}{4}g_{22}(F_{0\sigma}F^{0\sigma} + F_{1\sigma}F^{1\sigma}) \\
&= \frac{1}{4}g_{22}(F_{01}F^{01} + F_{10}F^{10}) \\
&= \frac{1}{2}g_{22}F_{01}F^{01} \\
&= \frac{1}{2}r^2f.(-fe^{-2\lambda-2\nu}) \\
&= -\frac{1}{2}r^2f^2e^{-2\lambda-2\nu}
\end{aligned}
\tag{A.131}
$$

$$\begin{aligned}
T_{33} &= -F_{3\rho}F_3{}^{\rho} + \frac{1}{4}g_{33}F_{\rho\sigma}F^{\rho\sigma} \\
&= \frac{1}{4}g_{33}F_{\rho\sigma}F^{\rho\sigma} \\
&= \frac{1}{4}g_{33}(F_{0\sigma}F^{0\sigma} + F_{1\sigma}F^{1\sigma}) \\
&= \frac{1}{4}g_{33}(F_{01}F^{01} + F_{10}F^{10}) \\
&= \frac{1}{2}g_{33}F_{01}F^{01} \\
&= \frac{1}{2}r^2\sin^2\theta f.(-fe^{-2\lambda-2\nu}) \\
&= -\frac{1}{2}r^2\sin^2\theta f^2e^{-2\lambda-2\nu} \\
&= \sin^2\theta T_{22} \qquad \text{(A.132)}
\end{aligned}$$

Dengan masing-masing komponen Tensor Ricci sama seperti solusi Schwarzschild

$$R_{00} = \left(-\nu'' + \nu'\lambda' - \nu'^2 - \frac{2\nu'}{r}\right)e^{(2\nu-2\lambda)} \qquad \text{(A.133)}$$

$$R_{11} = \nu'' + (\nu')^2 - \nu'\lambda' - \nu'\lambda' - \frac{2\lambda'}{r} \qquad \text{(A.134)}$$

$$R_{22} = (1 + r\nu' - r\lambda')\,e^{2\nu} - 1 \qquad \text{(A.135)}$$

Nilai-nilai ini disubstitusikan ke dalam persamaan medan Einstein (A.112)

$$R_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \qquad \text{(A.136)}$$

dengna nilai untuk komponen-00

$$\begin{aligned} R_{00} &= \frac{8\pi G}{c^4} T_{00} \\ \left(-\nu'' + \nu'\lambda' - \nu'^2 - \frac{2\nu'}{r}\right) e^{-2\lambda+2\nu} &= \frac{8\pi G}{c^4} \left(-\frac{1}{2} f^2 (e^{-2\lambda})\right) \end{aligned} \tag{A.137}$$

komponen-11

$$\begin{aligned} R_{11} &= \frac{8\pi G}{c^4} T_{11} \\ \left(\nu'' - \nu'\lambda' + \nu'^2 - \frac{2\lambda'}{r}\right) &= \frac{8\pi G}{c^4} \left(\frac{1}{2} f^2 e^{-2\nu}\right) \end{aligned} \tag{A.138}$$

dan komponen-22

$$\begin{aligned} R_{22} &= \frac{8\pi G}{c^4} T_{22} \\ (1 + r\nu' + r\lambda') e^{2\nu} - 1 &= \frac{8\pi G}{c^4} \left(-\frac{1}{2} r^2 f^2 e^{-2\lambda-2\nu}\right) \end{aligned} \tag{A.139}$$

pers.(A.137) dikalikan $e^{2\lambda}$ menjadi

$$\left(-\nu'' + \nu'\lambda' - \nu'^2 - \frac{2\nu'}{r}\right) e^{2\nu} = \frac{8\pi G}{c^4} \left(-\frac{1}{2} f^2\right) \tag{A.140}$$

pers.(A.138) dikalikan $e^{2\nu}$ menjadi

$$\left(\nu'' - \nu'\lambda' + \nu'^2 - \frac{2\lambda'}{r}\right) e^{2\nu} = \frac{8\pi G}{c^4} \left(\frac{1}{2} f^2\right) \tag{A.141}$$

kemudian pers.(A.137) dijumlahkan dengan pers.(A.138), sehingga

$$\begin{aligned} -\frac{2\nu'}{r}-\frac{2\lambda'}{r} &= 0 \\ \nu'+\lambda' &= 0 \\ \nu+\lambda &= \text{konstan} \end{aligned} \tag{A.142}$$

Pada $r \to \infty$, metrik kembali ke bentuk Minkowski sehingga $e^{2\nu} \to 1$ dan $e^{2\lambda} \to 1$, maka $\nu = 0$ dan $\lambda = 0$, sehingga

$$\begin{aligned} \nu+\lambda &= 0 \\ \nu &= -\lambda \end{aligned} \tag{A.143}$$

Dari pers.(A.139), maka

$$\begin{aligned} (1+r\nu'+r\lambda')\,e^{2\nu}-1 &= \frac{8\pi G}{c^4}\left(-\frac{1}{2}r^2f^2e^{-2\lambda-2\nu}\right) \\ (1+2r\nu')\,e^{2\nu}-1 &= \frac{8\pi G}{c^4}\left(-\frac{1}{2}r^2f^2e^{+2\nu-2\nu}\right) \\ \frac{d}{dr}\left(re^{2\nu}\right)-1 &= \frac{8\pi G}{c^4}\left(-\frac{1}{2}r^2f^2\right) \\ \frac{d}{dr}\left(re^{2\nu}\right) &= 1-\frac{4\pi G}{c^4}r^2f^2 \\ \frac{d}{dr}\left(re^{2\nu}\right) &= 1-\frac{4\pi G}{c^4}r^2\frac{q^2}{16\pi^2\epsilon_0r^4} \\ \frac{d}{dr}\left(re^{2\nu}\right) &= 1-\frac{Gq^2}{4\pi\epsilon_0r^2c^4} \\ re^{2\nu} &= \int\left(1-\frac{Gq^2}{4\pi\epsilon_0r^2c^4}\right)dr \\ re^{2\nu} &= \frac{Gq^2}{4\pi\epsilon_0c^4r}+r+C \\ e^{2\nu} &= \frac{Gq^2}{4\pi\epsilon_0c^4r^2}+1+\frac{C}{r} \end{aligned} \tag{A.144}$$

dengan $C = -2m$ (radius scwarschild) dan $\frac{Gq^2}{4\pi\epsilon_0 c^4} \equiv Q^2$, maka

$$e^{2\nu} = 1 + \frac{Q^2}{r^2} - \frac{2m}{r} \tag{A.145}$$

sehingga diperoleh solusi Reissner-Nordstrom

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right) du^2 + \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{A.146}$$

## A.7 Solusi Kerr

Untuk mempersingkat penulisan, diambil konstanta c=1.
Dari metrik Schwarzschild:

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{A.147}$$

Untuk menghilangkan singularitas di $r = 2m$, maka diperkenalkan koordinat waktu yang baru yaitu $cu = ct - r^*$ sehingga

$$\begin{aligned} t &= u + r^* \\ dt^2 &= du^2 + 2du\, dr^* + dr^{*2} \end{aligned} \tag{A.148}$$

$r^*$ adalah koordinat tortoise yang memenuhi hubungan

$$dr = \left(1 - \frac{2m}{r}\right) dr^* \tag{A.149}$$

kemudian pers. dan pers. dimasukkan ke dalam pers.1 sehingga

$$\begin{aligned} ds^2 &= -\left(1-\frac{2m}{r}\right)dt^2 \\ &\quad +\left(1-\frac{2m}{r}\right)^{-1}\left(1-\frac{2m}{r}\right)^2 dr^{*2}+r^2d\Omega^2 \\ &= -\left(1-\frac{2m}{r}\right)dt^2 \\ &\quad +\left(1-\frac{2m}{r}\right)dr^{*2}+r^2d\Omega^2 \\ &= -\left(1-\frac{2m}{r}\right)\left(dt^2-dr^{*2}\right)+r^2d\Omega^2 \\ &= -\left(1-\frac{2m}{r}\right)\left(du^2+2du\,dr^*+dr^{*2}-dr^{*2}\right)+r^2d\Omega^2 \\ &= -\left(1-\frac{2m}{r}\right)\left(du^2+2du\,dr^*\right)+r^2d\Omega^2 \\ &= -\left(1-\frac{2m}{r}\right)2du^2 \\ &\quad -\left(1-\frac{2m}{r}\right)2du\left(1-\frac{2m}{r}\right)^{-1}dr+r^2d\Omega^2 \\ &= -\left(1-\frac{2m}{r}\right)du^2-2du\,dr+r^2d\Omega^2 \end{aligned} \tag{A.150}$$

sehingga tensor metrik dari metrik di atas adalah:

$$g_{\mu\nu}=\begin{pmatrix} -\left(1-\frac{2m}{r}\right) & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\ \sin^2\theta \end{pmatrix} \tag{A.151}$$

bentuk kontravarian dari tensor metrik di atas dapat dicari dengan cara:

$$\begin{aligned}
g_{\mu\nu} g^{\mu\nu} &= 1 \\
g_{\mu\nu}^{-1} g_{\mu\nu} g^{\mu\nu} &= g_{\mu\nu}^{-1} \\
g^{\mu\nu} &= g_{\mu\nu}^{-1} \\
g^{\mu\nu} &= \frac{1}{|g_{\mu\nu}|} Adj\ \{g_{\mu\nu}\}
\end{aligned} \tag{A.152}$$

dengan $|g_{\mu\nu}|$ adalah:

$$\begin{aligned}
|g_{\mu\nu}| &= \begin{vmatrix} -\left(1-\frac{2m}{r}\right) & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \\ sin^2\theta & & & \end{vmatrix} \\
&= -\left(1-\frac{2m}{r}\right) \begin{vmatrix} 0 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \\ sin^2\theta & & \end{vmatrix} \\
&\quad -(-1) \begin{vmatrix} -1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \\ sin^2\theta & & \end{vmatrix} \\
&= 0+1 \begin{vmatrix} -1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{vmatrix} \\
&= -1 \begin{vmatrix} r^2 & 0 \\ 0 & r^2 \sin^2\theta \end{vmatrix} \\
&= -r^4 \sin^2\theta
\end{aligned} \tag{A.153}$$

Serta Adj$g_{\mu\nu}\}$ adalah sebagai berikut:
Untuk komponen-00, maka:

$$\begin{vmatrix} 0 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{vmatrix} = 0$$

Untuk komponen-01 = 10:

$$-\begin{vmatrix} -1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{vmatrix} = -1(-1)\begin{vmatrix} r^2 & 0 \\ 0 & r^2\sin^2\theta \end{vmatrix}$$
$$= r^4\sin^2\theta$$

Untuk komponen-02 = 20:

$$\begin{vmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{vmatrix} = 0$$

Untuk komponen-03 = 30:

$$-\begin{vmatrix} -1 & 0 & 0 \\ 0 & 0 & r^2 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

Untuk komponen-11:

$$\begin{vmatrix} -\left(1-\frac{2m}{r}\right) & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{vmatrix} = -\left(1-\frac{2m}{r}\right) r^4\sin^2\theta$$

Untuk komponen-12 = 21:

$$-\begin{vmatrix} -\left(1-\frac{2m}{r}\right) & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{vmatrix} = 0$$

Untuk komponen-13 = 31:

$$\begin{vmatrix} -\left(1-\frac{2m}{r}\right) & -1 & 0 \\ 0 & 0 & r^2 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

Untuk komponen-22:

$$\begin{vmatrix} -\left(1-\frac{2m}{r}\right) & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{vmatrix} = -\left(1-\frac{2m}{r}\right)\begin{vmatrix} 0 & 0 \\ 0 & r^2\sin^2\theta \end{vmatrix}$$

$$-(-1)\begin{vmatrix} -c & 0 \\ 0 & r^2\sin^2\theta \end{vmatrix}$$

$$= -r^2\sin^2\theta \qquad \text{(A.154)}$$

Untuk komponen-23 $=$ 32:

$$-\begin{vmatrix} -\left(1-\frac{2m}{r}\right) & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

Untuk komponen-33:

$$\begin{vmatrix} -\left(1-\frac{2m}{r}\right) & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & r^2 \end{vmatrix}$$

$$= -\left(1-\frac{2m}{r}\right)\begin{vmatrix} 0 & 0 \\ 0 & r^2 \end{vmatrix}$$

$$-(-1)\begin{vmatrix} -1 & 0 \\ 0 & r^\theta \end{vmatrix}$$

$$= -r^2$$

Sehingga tensor metrik kontravarian $g^{\mu\nu}$ menjadi:

$$\begin{aligned} g^{\mu\nu} &= \frac{1}{|g_{\mu\nu}|}\text{Adj}\{g^{\mu\nu}\} \\ &= \frac{1}{-r^4\sin^2\theta} \\ &\begin{pmatrix} 0 & r^4\sin^2\theta & 0 & 0 \\ r^4\sin^2\theta & \left(1-\frac{2m}{r}\right)r^4\sin^2\theta & 0 & 0 \\ 0 & 0 & -r^2\sin^2\theta & 0 \\ 0 & 0 & 0 & -r^2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & \left(1-\frac{2m}{r}\right) & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix} \end{aligned} \tag{A.155}$$

karena elemen garis dalam pernyataan tensor metrik adalah

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu \\ &= -l_\mu n_\nu - l_\nu n_\mu + m_\mu\bar{m}_\nu + m_\nu\bar{m}_\mu \end{aligned} \tag{A.156}$$

kemudian dilakukan $\nu \leftrightarrow \mu$ pada suku ke-2 dan ke-4 sehingga:

$$ds^2 = -2l_\mu n_\nu + 2m_\mu\bar{m}_\nu \tag{A.157}$$

Kemudian dipilih suku-suku vektor-4 null dari metrik Reissner Nordstrom diatas. Pemilihan ini bebas asalkan memenuhi sifat-sifat vektor-4 null yang telah didefinisikan. Dari metrik Reissner-Nordstrom:

$$\begin{aligned} ds^2 &= -\left(1-\frac{2m}{r}\right)du^2 - 2du\,dr + r^2d\Omega^2 \\ &= -\left[\left(1-\frac{2m}{r}\right)du + 2dr\right]du \\ &\quad +r\left(d\theta + i\ \sin\theta\,d\phi\right)r\left(d\theta - i\ \sin\theta\,d\phi\right) \end{aligned} \tag{A.158}$$

sehingga pemilihan komponen vektor-4 null adalah:

$$l_\mu dx^\mu = du \tag{A.159}$$

sehingga $l_\mu = (1, 0, 0, 0)$.
Kemudian untuk komponen $n_\mu$:

$$\begin{aligned} -2n_\mu dx^\mu &= -\left(1 - \frac{2m}{r}\right) du + 2dr \\ n_\mu dx^\mu &= \frac{1}{2}\left(1 - \frac{2m}{r}\right) du + dr \end{aligned} \tag{A.160}$$

sehingga $n_\mu = \left(\frac{1}{2}\left(1 - \frac{2m}{r}\right), 1, 0, 0\right)$.
Untuk komponen $m_\mu$:

$$\begin{aligned} \sqrt{2} m_\mu dx^\mu &= r\left(d\theta + i\ \sin\theta\ d\phi\right) \\ &= \frac{r}{\sqrt{2}}\left(d\theta + i\ \sin\theta\ d\phi\right) \end{aligned} \tag{A.161}$$

Sehingga $m_\mu = (0, 0, 1, i\ \sin\theta)$.
$\bar{m}_\mu$ adalah kompleks konjugat dari $m_\mu$ sehingga $\bar{m}_\mu = (0, 0, 1, -i\ \sin\theta)$.

Maka kompenen-komponen vektor-4 null yang telah dipilih adalah:

$$\begin{aligned} l_\mu &= (1, 0, 0, 0) \\ n_\mu &= \left(\frac{1}{2}\left(1 - \frac{2m}{r}\right), 1, 0, 0\right) \\ m_\mu &= (0, 0, 1, i\ \sin\theta) \\ \bar{m}_\mu &= (0, 0, 1, -i\ \sin\theta) \end{aligned} \tag{A.162}$$

bentuk kontravarian dari masing-masing komponen vektor-4 null di atas dapat dicari menggunakan tensor metrik kontravarian:

$$Z^\mu = g^{\mu\nu} Z_\nu \tag{A.163}$$

sehingga

$$l^\mu = g^{\mu\nu} l_\nu \tag{A.164}$$

nilai untuk masing-masing $\mu = (0, 1, 2, 3)$ adalah:
untuk $\mu = 0$

$$\begin{aligned} l^0 &= g^{0\nu} l_\nu \\ &= g^{00} l_0 + g^{01} l_1 + g^{02} l_2 + g^{03} l_3 \\ &= 0 \end{aligned}$$

untuk $\mu = 1$

$$\begin{aligned} l^1 &= g^{1\nu} l_\nu \\ &= g^{10} l_0 + g^{11} l_1 + g^{12} l_2 + g^{13} l_3 \\ &= -1 + 0 + 0 + 0 \\ &= 1 \end{aligned}$$

untuk $\mu = 2$

$$\begin{aligned} l^2 &= g^{2\nu} l_\nu \\ &= g^{20} l_0 + g^{21} l_1 + g^{22} l_2 + g^{23} l_3 \\ &= 0 \end{aligned}$$

untuk $\mu = 3$

$$\begin{aligned} l^3 &= g^{3\nu} l_\nu \\ &= g^{30} l_0 + g^{31} l_1 + g^{32} l_2 + g^{33} l_3 \\ &= 0 \end{aligned}$$

sehingga didapatkan:

$$l^\mu = (0, -1, 0, 0) \tag{A.165}$$

Kemudian untuk $n^\mu$ dengan masing-masing $\mu = (0, 1, 2, 3)$ adalah:

$$n^\mu = g^{\mu\nu} n_\nu \tag{A.166}$$

untuk $\mu = 0$

$$\begin{aligned} n^0 &= g^{0\nu} n_\nu \\ &= g^{00} n_0 + g^{01} n_1 + g^{02} n_2 + g^{03} n_3 \\ &= 0 + (-1).1 + 0 + 0 \\ &= -1 \end{aligned} \tag{A.167}$$

untuk $\mu = 1$

$$\begin{aligned} n^1 &= g^{1\nu} n_\nu \\ &= g^{10} n_0 + g^{11} n_1 + g^{12} n_2 + g^{13} n_3 \\ &= (-1)\frac{1}{2}\left(1 - \frac{2m}{r}\right) + \left(1 - \frac{2m}{r}\right).1 + 0 + 0 \\ &= \frac{1}{2}\left(1 - \frac{2m}{r}\right) \end{aligned} \tag{A.168}$$

untuk $\mu = 2$

$$\begin{aligned} n^2 &= g^{2\nu} n_\nu \\ &= g^{20} n_0 + g^{21} n_1 + g^{22} n_2 + g^{23} n_3 \\ &= 0 \end{aligned} \tag{A.169}$$

untuk $\mu = 3$

$$\begin{aligned} n^3 &= g^{3\nu} n_\nu \\ &= g^{30} n_0 + g^{31} n_1 + g^{32} n_2 + g^{33} n_3 \\ &= 0 \end{aligned} \tag{A.170}$$

untuk $\mu = 4$

$$\begin{aligned} n^4 &= g^{4\nu} n_\nu \\ &= g^{40} n_0 + g^{41} n_1 + g^{42} n_2 + g^{43} n_3 \\ &= 0 \end{aligned} \tag{A.171}$$

sehingga didapatkan:

$$n^\mu = \left(-1, \frac{1}{2}\left(1 - \frac{2m}{r}\right), 0, 0\right) \tag{A.172}$$

Kemudian untuk $m^\mu$ dengan masing-masing $\mu = (0, 1, 2, 3)$ adalah:

$$m^\mu = g^{\mu\nu} m_\nu \tag{A.173}$$

untuk $\mu = 0$

$$\begin{aligned} m^0 &= g^{0\nu} m_\nu \\ &= g^{00} m_0 + g^{01} m_1 + g^{02} m_2 + g^{03} m_3 \\ &= 0 \end{aligned}$$

untuk $\mu = 1$

$$\begin{aligned} m^1 &= g^{1\nu} m_\nu \\ &= g^{10} m_0 + g^{11} m_1 + g^{12} m_2 + g^{13} m_3 \\ &= 0 \end{aligned}$$

untuk $\mu = 2$

$$\begin{aligned} m^2 &= g^{2\nu} m_\nu \\ &= g^{20} m_0 + g^{21} m_1 + g^{22} m_2 + g^{23} m_3 \\ &= 0 + 0 + \frac{1}{r^2} \frac{r}{\sqrt{2}} + 0 \\ &= \frac{1}{r\sqrt{2}} \end{aligned}$$

untuk $\mu = 3$

$$\begin{aligned}
m^3 &= g^{3\nu} m_\nu \\
&= g^{30} m_0 + g^{31} m_1 + g^{32} m_2 + g^{33} m_3 \\
&= 0 + 0 + 0 + \frac{1}{r^2 \sin^2\theta} \frac{ir \ \sin\theta}{\sqrt{2}} \\
&= \frac{i}{r\sqrt{2}\sin\theta}
\end{aligned}$$

sehingga didapatkan:

$$m^\mu = \frac{1}{r\sqrt{2}}(0, 0, 1, \frac{i}{\sin\theta}) \tag{A.174}$$

$\bar{m}$ adalah kompleks ku=onjugat dari $m$ sehingga:

$$\bar{m}^\mu = \frac{1}{r\sqrt{2}}(0, 0, 1, -\frac{i}{\sin\theta}) \tag{A.175}$$

Maka bentuk kontravarian dari komponen vektor-4 null adalah

$$\begin{aligned}
l^\mu &= (0, -1, 0, 0) \\
n^\mu &= (-1, \frac{1}{2}\left(1 - \frac{2m}{r}\right), 0, 0) \\
m^\mu &= \frac{1}{r\sqrt{2}}(0, 0, 1, \frac{i}{\sin\theta}) \\
\bar{m}^\mu &= \frac{1}{r\sqrt{2}}(0, 0, 1, \frac{i}{\sin\theta})
\end{aligned} \tag{A.176}$$

atau

$$\begin{aligned}
l^\mu\partial_\mu &= -\partial_r \\
n^\mu\partial_\mu &= -\partial_u + \frac{1}{2}\left(1-\frac{2m}{r}\right)\partial_r \\
m^\mu\partial_\mu &= \frac{1}{r\sqrt{2}}\left(\partial_\theta + \frac{i}{\sin\theta}\partial_\phi\right) \\
m^\mu\partial_\mu &= \frac{1}{r\sqrt{2}}\left(\partial_\theta - \frac{i}{\sin\theta}\partial_\phi\right)
\end{aligned} \tag{A.177}$$

Kemudian nilai $r$ dijadikan dalam bentuk kompleks sehingga:

$$\begin{aligned}
l^\mu &= (0,-1,0,0) \\
n^\mu &= (-1,\frac{1}{2}\left(1-\frac{m}{r}-\frac{m}{\bar{r}}\right),0,0) \\
m^\mu &= \frac{1}{\bar{r}\sqrt{2}}(0,0,1,\frac{i}{\sin\theta}) \\
\bar{m}^\mu &= \frac{1}{r\sqrt{2}}(0,0,1,\frac{i}{\sin\theta})
\end{aligned} \tag{A.178}$$

atau

$$\begin{aligned}
l^\mu\partial_\mu &= -\partial_r \\
n^\mu\partial_\mu &= -\partial_u + \frac{1}{2}\left(1-\frac{m}{r}-\frac{m}{\bar{r}}\right)\partial_r \\
m^\mu\partial_\mu &= \frac{1}{\bar{r}\sqrt{2}}\left(\partial_\theta + \frac{i}{\sin\theta}\partial_\phi\right) \\
\bar{m}^\mu\partial_\mu &= \frac{1}{r\sqrt{2}}\left(\partial_\theta - \frac{i}{sin\theta}\partial_\phi\right)
\end{aligned} \tag{A.179}$$

setelah itu dilakukan transformasi

$$x^\rho \to x'^\rho = x^\rho + ia\cos\theta(\delta_0^\rho - \delta_1^\rho) \tag{A.180}$$

dengan a adalah suatu konstanta yang akan ditentukan kemudian sehingga:

$$\begin{aligned}
u \to u' &= u + ia\cos\theta \\
r \to r' &= r - ia\cos\theta \\
\theta \to \theta' &= \theta \\
\phi \to \phi' &= \phi
\end{aligned} \tag{A.181}$$

atau

$$\begin{aligned}
u &= u' - ia\cos\theta = u' - ia\cos\theta' \\
r &= r' + ia\cos\theta = r' + ia\cos\theta'
\end{aligned} \tag{A.182}$$

dengan $v', r', \theta', \phi'$ adalah kuantitas riil. Vektor-vektor basis bertransformasi dengan menggunakan aturan rantai:

$$\frac{\partial}{\partial x^\mu} \to \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu}$$

atau dapat dituliskan :

$$\partial_\mu \to \partial_{\mu'} = \frac{\partial x^\nu}{\partial x'^\mu}\partial_\nu \tag{A.183}$$

sehingga untuk masing-masing $\mu = (0, 1, 2, 3)$ didapatkan:

$$\begin{aligned}
\partial_0 \to \partial_{0'} &= \frac{\partial x^\nu}{\partial x'^0}\partial_\nu \\
\partial_{u'} &= \frac{\partial x^\nu}{\partial u'}\partial_\nu \\
&= \frac{\partial u}{\partial u'}\partial_u + \frac{\partial r}{\partial u'}\partial_r + \frac{\partial \theta}{\partial u'}\partial_\theta + \frac{\partial \phi}{\partial u'}\partial_\phi \\
&= \partial_u
\end{aligned} \tag{A.184}$$

$$\begin{aligned}
\partial_1 \rightarrow \partial_{1'} &= \frac{\partial x^{\nu}}{\partial x'^{1}}\partial_{\nu} \\
\partial_{r'} &= \frac{\partial x^{\nu}}{\partial r'}\partial_{\nu} \\
&= \frac{\partial u}{\partial r'}\partial_u + \frac{\partial r}{\partial r'}\partial_r + \frac{\partial \theta}{\partial r'}\partial_\theta + \frac{\partial \phi}{\partial r'}\partial_\phi \\
&= \partial_r \qquad \text{(A.185)}
\end{aligned}$$

$$\begin{aligned}
\partial_2 \rightarrow \partial_{2'} &= \frac{\partial x^{\nu}}{\partial x'^{2}}\partial_{\nu} \\
\partial_{\theta'} &= \frac{\partial x^{\nu}}{\partial \theta'}\partial_{\nu} \\
&= \frac{\partial u}{\partial \theta'}\partial_u + \frac{\partial r}{\partial \theta'}\partial_r + \frac{\partial \theta}{\partial \theta'}\partial_\theta + \frac{\partial \phi}{\partial \theta'}\partial_\phi \\
&= ia\sin\theta'\ \partial_u - ia\sin\theta'\ \partial_r + \partial_\theta \qquad \text{(A.186)}
\end{aligned}$$

$$\begin{aligned}
\partial_3 \rightarrow \partial_{3'} &= \frac{\partial x^{\nu}}{\partial x^{3}}\partial_{\nu} \\
\partial_{\phi'} &= \frac{\partial x^{\nu}}{\partial \phi'}\partial_{\nu} \\
&= \frac{\partial u}{\partial \phi'}\partial_u + \frac{\partial r}{\partial \phi'}\partial_r + \frac{\partial \theta}{\partial \phi'}\partial_\theta + \frac{\partial \phi}{\partial \phi'}\partial_\phi \\
&= \partial_\phi \qquad \text{(A.187)}
\end{aligned}$$

Maka komponen-komponen vektor-4 null menjadi:

$$l^{\mu}\partial_{\mu} = -\partial_r \rightarrow l'^{\mu}\partial_{\mu'} = -\partial_{r'} = -\partial_r \qquad \text{(A.188)}$$

$$
\begin{aligned}
n^\mu \partial_\mu \to n'^\mu \partial_{\mu'} &= -\partial_{u'} + \frac{1}{2}\left(1 - \frac{m}{r'} - \frac{m}{\bar{r}'}\right)\partial_{r'} \\
&= -\partial_u + \frac{1}{2}\left(1 - \frac{m}{r' + ia\cos\theta'} - \frac{m}{r' - ia\cos\theta'}\right)\partial_{r'} \\
&= -\partial_u + \frac{1}{2}\left(1 - \frac{2mr'}{r'^2 + a^2\cos^2\theta'}\right)\partial_r \qquad \text{(A.189)}
\end{aligned}
$$

$$
\begin{aligned}
m^\mu \partial_\mu \to m'^\mu \partial_{\mu'} &= \frac{1}{\bar{r}\sqrt{2}}\left(\partial_{\theta'} + \frac{i}{\sin\theta}\partial_{\phi'}\right) \\
&= \frac{1}{\sqrt{2}(r' - ia\cos\theta')}\,(ia\sin\theta'\partial_{v'} - ia\sin\theta'\partial_{r'} \\
&\quad +\partial_{\theta'} + \frac{i}{\sin\theta'}\partial_{\phi'}) \qquad \text{(A.190)}
\end{aligned}
$$

$$
\begin{aligned}
\bar{m}^\mu \partial_\mu \to \bar{m}'^\mu \partial_{\mu'} &= \frac{1}{r\sqrt{2}}\left(\partial_\theta - \frac{i}{\sin\theta}\partial_\phi\right) \\
&= \frac{1}{\sqrt{2}(r' + ia\cos\theta')}\,(-ia\sin\theta'\partial_{v'} + ia\sin\theta'\partial_{r'} \\
&\quad +\partial_{\theta'} - \frac{i}{\sin\theta'}\partial_{\phi'}) \qquad \text{(A.191)}
\end{aligned}
$$

dengan menghilangkan tanda $'$ pada persamaan-persamaan di atas, maka diperoleh:

$$\begin{aligned}
l^\mu \partial_\mu &= -\partial_r \\
n^\mu \partial_\mu &= -\partial_u + \frac{1}{2}\left(1 - \frac{2mr}{r^2 + a^2\cos^2\theta}\right)\partial_r \\
m^\mu \partial_\mu &= \frac{1}{\sqrt{2}(r - ia\cos\theta)}\left(ia\sin\theta\partial_v - ia\sin\theta\partial_r + \partial_\theta + \frac{i}{\sin\theta}\partial_\phi\right) \\
\bar{m}^\mu \partial_\mu &= \frac{1}{\sqrt{2}(r + ia\cos\theta)}\left(-ia\sin\theta\partial_v + ia\sin\theta\partial_r + \partial_\theta - \frac{i}{\sin\theta}\partial_\phi\right)
\end{aligned} \tag{A.192}$$

atau

$$\begin{aligned}
l^\mu &= (0, -1, 0, 0) \\
n^\mu &= (-1, \frac{1}{2}\left(1 - \frac{2mr}{r^2 + a^2\cos^2\theta}\right), 0, 0) \\
m^\mu &= \frac{1}{\sqrt{2}(r - ia\cos\theta)}\left(ia\ \sin\theta, -ia\ \sin\theta, 1, \frac{i}{\sin\theta}\right) \\
\bar{m}^\mu &= \frac{1}{\sqrt{2}(r + ia\ cos\theta)}\left(-ia\sin\theta, ia\sin\theta, 1, -\frac{i}{\sin\theta}\right)
\end{aligned} \tag{A.193}$$

Selanjutnya komponen-komponen tensor metrik kontravarian $g^{\mu\nu}$ dapat dibentuk yaitu:

$$g^{\mu\nu} = -l^\mu n^\nu - l^\mu n^\nu + m^\mu \bar{m}^\nu + m^\nu \bar{m}^\mu \tag{A.194}$$

masing-masing komponen tersebut adalah

$$\begin{aligned}
g^{00} &= -l^0 n^0 - l^0 n^0 + m^0 \bar{m}^0 + m^0 \bar{m}^0 \\
&= -2l^0 n^0 + 2m^0 \bar{m}^0 \\
&= 0 + 2\frac{ia\sin\theta}{\sqrt{2}(r - ia\cos\theta)} . \frac{-ia\ \sin\theta}{\sqrt{2}(r + ia\cos\theta)} \\
&= \frac{a^2\sin^2\theta}{(r^2 + a^2\cos^2\theta)} \\
&= \frac{a^2\sin^2\theta}{\rho^2} \qquad \text{(A.195)}
\end{aligned}$$

dengan

$$\rho^2 \equiv r^2 + a^2\cos^2\theta \qquad \text{(A.196)}$$

$$\begin{aligned}
g^{01} &= g^{10} \\
&= -l^0 n^1 - l^1 n^0 + m^0 \bar{m}^1 + m^1 \bar{m}^0 \\
&= 0 - (-1).(-1) \\
&\quad + \frac{1}{\sqrt{2}(r - ia\cos\theta)}(ia\sin\theta) . \frac{1}{\sqrt{2}(r + ia\cos\theta)}(ia\sin\theta) \\
&\quad + \frac{1}{\sqrt{2}(r + ia\cos\theta)}(-ia\sin\theta) . \frac{1}{\sqrt{2}(r - ia\cos\theta)}(-ia\sin\theta) \\
&= -\left(1 + \frac{a^2\sin^2\theta}{r^2 + a^2\cos^2\theta}\right) \\
&= -\left(\frac{r^2 + a^2\cos^2\theta + a^2\sin^2\theta}{r^2 + a^2\cos^2\theta}\right) \\
&= -\left(\frac{r^2 + a^2}{r^2 + a^2\cos^2\theta}\right) \\
&= -\left(\frac{r^2 + a^2}{\rho^2}\right) \qquad \text{(A.197)}
\end{aligned}$$

$$\begin{aligned}
g^{02} &= g^{20} \\
&= -l^0 n^2 - l^2 n^0 + m^0 \bar{m}^2 + m^2 \bar{m}^0 \\
&= 0 + \frac{ia \sin\theta}{\sqrt{2}(r - ia\cos\theta)} . \frac{1}{\sqrt{2}(r + ia\cos\theta)} \\
&\quad + \frac{1}{\sqrt{2}(r - ia\cos\theta)} . \frac{-ia\sin\theta}{\sqrt{2}(r + ia\cos\theta)} \\
&= 0 \qquad \text{(A.198)}
\end{aligned}$$

$$\begin{aligned}
g^{03} &= g^{30} \\
&= -l^0 n^3 - l^3 n^0 + m^0 \bar{m}^3 + m^3 \bar{m}^0 \\
&= 0 + \frac{ia \sin\theta}{\sqrt{2}(r - ia\cos\theta)} . \frac{-i}{\sqrt{2}\sin\theta(r + ia\cos\theta)} \\
&\quad + \frac{i}{\sqrt{2}\sin\theta(r - ia\cos\theta)} . \frac{-ia\sin\theta}{\sqrt{2}(r + ia\cos\theta)} \\
&= \frac{a}{r^2 + a^2\cos^2\theta} \\
&= \frac{a}{\rho^2} \qquad \text{(A.199)}
\end{aligned}$$

$$\begin{aligned}
g^{11} &= -l^1 n^1 - l^1 n^1 + m^1 \bar{m}^1 + m^1 \bar{m}^1 \\
&= -2l^1 n^1 + 2m^1 \bar{m}^1 + \\
&= -2.(-1).\frac{1}{2}\left(1 - \frac{2mr}{r^2 + a^2\cos^2\theta}\right) \\
&\quad +2.\frac{-ia\sin\theta}{\sqrt{2}(r - ia\cos\theta)}.\frac{ia\sin\theta}{\sqrt{2}(r + ia\cos\theta)} \\
&= 1 - \frac{2mr}{r^2 + a^2\cos^2\theta} + \frac{a^2\sin^2\theta}{r^2 + a^2\cos^2\theta} \\
&= 1 - \frac{2mr}{\rho^2} + \frac{a^2\sin^2\theta}{\rho^2} \\
&= \frac{\rho^2 - 2mr + Q^2 + a^2\sin^2\theta}{\rho^2} \\
&= \frac{r^2 + a^2\cos^2\theta - 2mr + a^2\sin^2\theta}{\rho^2} \\
&= \frac{r^2 + a^2 - 2mr}{\rho^2}
\end{aligned} \tag{A.200}$$

$$\begin{aligned}
g^{12} &= g^{21} \\
&= -l^1 n^2 - l^2 n^1 + m^1 \bar{m}^2 + m^2 \bar{m}^1 \\
&= 0 + \frac{-ia\ \sin\theta}{\sqrt{2}(r - ia\ \cos\theta)}.\frac{1}{\sqrt{2}\sin\theta(r + ia\ \cos\theta)} \\
&\quad + \frac{1}{\sqrt{2}\sin\theta(r - ia\ \cos\theta)}.\frac{ia\ \sin\theta}{\sqrt{2}(r + ia\ \cos\theta)} \\
&= 0
\end{aligned} \tag{A.201}$$

$$
\begin{aligned}
g^{13} &= g^{31} \\
&= -l^1 n^3 - l^3 n^1 + m^1 \bar{m}^3 + m^3 \bar{m}^1 \\
&= 0 + \frac{-ia\ \sin\theta}{\sqrt{2}(r - ia\ \cos\theta)} . \frac{-i}{\sqrt{2}\sin\theta(r + ia\ \cos\theta)} \\
&\quad + \frac{i}{\sqrt{2}\sin\theta(r - ia\ \cos\theta)} . \frac{ia\ \sin\theta}{\sqrt{2}(r + ia\ \cos\theta)} \\
&= -\frac{a}{r^2 + a^2\cos^2\theta} \\
&= -\frac{a}{\rho^2} \qquad \text{(A.202)}
\end{aligned}
$$

$$
\begin{aligned}
g^{22} &= -l^2 n^2 - l^2 n^2 + m^2 \bar{m}^2 + m^2 \bar{m}^2 \\
&= -2l^2 n^2 + 2m^2 \bar{m}^2 \\
&= 0 + 2\frac{1}{\sqrt{2}(r - ia\ \cos\theta)} . \frac{1}{\sqrt{2}\sin\theta(r + ia\ \cos\theta)} \\
&= \frac{1}{r^2 + a^2\cos^2\theta} \\
&= \frac{1}{\rho^2} \qquad \text{(A.203)}
\end{aligned}
$$

$$
\begin{aligned}
g^{23} &= g^{32} \\
&= -l^2 n^3 - l^3 n^2 + m^2 \bar{m}^3 + m^3 \bar{m}^2 \\
&= 0 + \frac{1}{\sqrt{2}(r - ia\ \cos\theta)} . \frac{-i}{\sqrt{2}\sin\theta(r + ia\ \cos\theta)} \\
&\quad + \frac{i}{\sqrt{2}\sin\theta(r - ia\ \cos\theta)} . \frac{1}{\sqrt{2}(r + ia\ \cos\theta)} \\
&= 0 \qquad \text{(A.204)}
\end{aligned}
$$

$$\begin{aligned}
g^{33} &= -l^3n^3 - l^3n^3 + m^3\bar{m}^3 + m^3\bar{m}^3 \\
&= -2l^3n^3 + 2m^3\bar{m}^3 \\
&= 0 + 2\frac{i}{\sqrt{2}\sin\theta(r - ia\ \cos\theta)}.\frac{-i}{\sqrt{2}\sin\theta(r + ia\ \cos\theta)} \\
&= \frac{1}{\sin^2\theta(r^2 + a^2\cos^2\theta)} \\
&= \frac{1}{\rho^2\sin^2\theta}
\end{aligned} \tag{A.205}$$

Sehingga tensor metrik kontravarian adalah

$$g^{\mu\nu} = \begin{pmatrix} \frac{a^2\ \sin^2\theta}{\rho^2} & -\left(\frac{r^2+a^2}{\rho^2}\right) & 0 & \frac{a}{\rho^2} \\ -\left(\frac{r^2+a^2}{\rho^2}\right) & \frac{r^2+a^2-2mr}{\rho^2} & 0 & -\frac{a}{\rho^2} \\ 0 & 0 & \frac{1}{\rho^2} & 0 \\ \frac{a}{\rho^2} & -\frac{a}{\rho^2} & 0 & \frac{1}{\rho^2\sin^2\theta} \end{pmatrix} \tag{A.206}$$

karena $\rho^2 \equiv r^2 + a^2\cos^2\theta$ , maka:

$$g^{\mu\nu} = \begin{pmatrix} \frac{a^2\ \sin^2\theta}{\rho^2} & -\left(\frac{\rho^2+a^2\sin^2\theta}{\rho^2}\right) & 0 & \frac{a}{\rho^2} \\ -\left(\frac{\rho^2+a^2\sin^2\theta}{\rho^2}\right) & \frac{r^2+a^2-2mr}{\rho^2} & 0 & -\frac{a}{\rho^2} \\ 0 & 0 & \frac{1}{\rho^2} & 0 \\ \frac{a}{\rho^2} & -\frac{a}{\rho^2} & 0 & \frac{1}{\rho^2\sin^2\theta} \end{pmatrix} \tag{A.207}$$

Bentuk kovarian dari tensor metrik di atas adalah

$$g_{\mu\nu} = \frac{1}{|g^{\mu\nu}|}\text{Adj}\ \{g^{\mu\nu}\} \tag{A.208}$$

dengan

$$
\begin{aligned}
|g^{\mu\nu}| &= \begin{vmatrix} \frac{a^2\ \sin^2\theta}{\rho^2} & -\left(\frac{\rho^2+a^2\sin^2\theta}{\rho^2}\right) & 0 & \frac{a}{\rho^2} \\ -\left(\frac{\rho^2+a^2\sin^2\theta}{\rho^2}\right) & \frac{r^2+a^2-2mr}{\rho^2} & 0 & -\frac{a}{\rho^2} \\ 0 & 0 & \frac{1}{\rho^2} & 0 \\ \frac{a}{\rho^2} & -\frac{a}{\rho^2} & 0 & \frac{1}{\rho^2\sin^2\theta} \end{vmatrix} \\
&= \frac{a^2\ \sin^2\theta}{\rho^2}\left[\frac{r^2+a^2-2mr}{\rho^2}\left(\frac{1}{\rho^4\sin^2\theta}\right)\right] \\
&\quad +\frac{a^2\ \sin^2\theta}{\rho^2}\left(-\frac{a}{\rho^2}\right)\frac{a}{\rho^4} \\
&\quad +\frac{\rho^2+a^2\ \sin^2\theta}{\rho^2}\left[-\frac{\rho^2 a^2\ \sin^2\theta}{\rho^2}\left(\frac{1}{\rho^4\sin^2\theta}\right)\right] \\
&\quad +\frac{\rho^2+a^2\ \sin^2\theta}{\rho^2}\left[\left(-\frac{a}{\rho^2}\right)\frac{a}{\rho^4}\right] \\
&\quad +\left(-\frac{a}{\rho^2}\right)\left[-\frac{\rho^2+a^2\ \sin^2\theta}{\rho^2}\left(\frac{a}{\rho^4}\right)\right] \\
&\quad +\left(-\frac{a}{\rho^2}\right)\left[-\frac{r^2+a^2-2mr}{\rho^2}\left(\frac{a}{\rho^4}\right)\right] \\
&= -\frac{1}{\rho^4\sin^2\theta} \qquad \text{(A.209)}
\end{aligned}
$$

Serta komponen-komponen Adj $\{g^{\mu\nu}\}$ adalah:

Untuk komponen-00:

$$
\begin{align}
\begin{vmatrix} \frac{r^2+a^2-2mr}{\rho^2} & 0 & -\frac{a}{\rho^2} \\ 0 & \frac{1}{\rho^2} & 0 \\ -\frac{a}{\rho^2} & 0 & \frac{1}{\rho^2\sin^2\theta} \end{vmatrix} &= \frac{r^2+a^2-2mr}{\rho^2} \begin{vmatrix} \frac{1}{\rho^2} & 0 \\ 0 & \frac{1}{\rho^2\sin^2\theta} \end{vmatrix} \\
&+ \left(-\frac{a}{\rho^2}\right) \begin{vmatrix} 0 & \frac{1}{\rho^2} \\ -\frac{a}{\rho^2} & 0 \end{vmatrix} \\
&= \frac{r^2+a^2-2mr}{\rho^2} \frac{1}{\rho^2} \frac{1}{\rho^2\sin^2\theta} \\
&+ \left(-\frac{a}{\rho^2}\right)\left(\frac{1}{\rho^2}\frac{a}{\rho^2}\right) \\
&= \frac{r^2+a^2-2mr-a^2\sin^2\theta}{\rho^6\sin^2\theta} \\
&= \frac{\rho^2-2mr}{\rho^6\sin^2\theta} \tag{A.210}
\end{align}
$$

Untuk komponen-01=komponen-10:

$$
\begin{align}
-\begin{vmatrix} -\frac{\rho^2+a^2\sin^2\theta}{\rho^2} & 0 & -\frac{a}{\rho^2} \\ 0 & \frac{1}{\rho^2} & 0 \\ \frac{a}{\rho^2} & 0 & \frac{1}{\rho^2\sin^2\theta} \end{vmatrix} &= \frac{\rho^2+a^2\sin^2\theta}{\rho^2} \begin{vmatrix} \frac{1}{\rho^2} & 0 \\ 0 & \frac{1}{\rho^2\sin^2\theta} \end{vmatrix} \\
& \frac{a}{\rho^2} \begin{vmatrix} 0 & \frac{1}{\rho^2} \\ \frac{a}{\rho^2} & 0 \end{vmatrix} \\
&= \frac{\rho^2+a^2\sin^2\theta}{\rho^2} \frac{1}{\rho^4\sin^2\theta} - \frac{a^2}{\rho^6} \\
&= \frac{1}{\rho^4\sin^2\theta} \tag{A.211}
\end{align}
$$

Untuk komponen-02=komponen-20:

$$\begin{vmatrix} -\frac{\rho^2+a^2\sin^2\theta}{\rho^2} & \frac{r^2+a^2-2mr}{\rho^2} & -\frac{a}{\rho^2} \\ 0 & 0 & 0 \\ \frac{a}{\rho^2} & \frac{a}{\rho^2} & \frac{1}{\rho^2\sin^2\theta} \end{vmatrix} = 0 \tag{A.212}$$

Untuk komponen-03=komponen-30:

$$\begin{aligned} -\begin{vmatrix} -\frac{\rho^2+a^2\sin^2\theta}{\rho^2} & \frac{r^2+a^2-2mr}{\rho^2} & 0 \\ 0 & 0 & \frac{1}{\rho^2} \\ \frac{a}{\rho^2} & -\frac{a}{\rho^2} & 0 \end{vmatrix} &= \frac{\rho^2+a^2\sin^2\theta}{\rho^2}\begin{vmatrix} 0 & \frac{1}{\rho^2} \\ -\frac{a}{\rho^2} & 0 \end{vmatrix} \\ &\quad + \frac{r^2+a^2-2mr}{\rho^2}\left(-\frac{a}{\rho^4}\right) \\ &= a\frac{\rho^2+a^2\sin^2\theta}{\rho^6} \\ &\quad -a\frac{r^2+a^2-2mr}{\rho^6} \\ &= a\frac{r^2+a^2}{\rho^6} \\ &\quad -a\frac{r^2+a^2-2mr}{\rho^6} \\ &= \frac{2mra}{\rho^6} \end{aligned} \tag{A.213}$$

Untuk komponen-11:

$$\begin{vmatrix} \frac{a^2\sin^2\theta}{\rho^2} & 0 & \frac{a}{\rho^2} \\ 0 & \frac{1}{\rho^2} & 0 \\ \frac{a}{\rho^2} & 0 & \frac{1}{\rho^2\sin^2\theta} \end{vmatrix} = \frac{a^2}{\rho^6} - \frac{a^2}{\rho^6} = 0 \tag{A.214}$$

Untuk komponen-12=komponen-21:

$$-\begin{vmatrix} \frac{a^2\sin^2\theta}{\rho^2} & -\frac{\rho^2+a^2\sin^2\theta}{\rho^2} & \frac{a}{\rho^2} \\ 0 & 0 & 0 \\ \frac{a}{\rho^2} & -\frac{a}{\rho^2} & \frac{1}{\rho^2\sin^2\theta} \end{vmatrix} = 0 \qquad \text{(A.215)}$$

Untuk komponen-13=komponen-31:

$$\begin{aligned} \begin{vmatrix} \frac{a^2\sin^2\theta}{\rho^2} & -\frac{\rho^2+a^2\sin^2\theta}{\rho^2} & 0 \\ 0 & 0 & \frac{1}{\rho^2} \\ \frac{a}{\rho^2} & -\frac{a}{\rho^2} & 0 \end{vmatrix} &= \frac{a^3\sin^2\theta}{\rho^6} \\ &\quad -a\frac{\rho^2+a^2\sin^2\theta}{\rho^6} \\ &= -\frac{a}{\rho^4} \qquad \text{(A.216)} \end{aligned}$$

Untuk komponen-22;

$$\begin{vmatrix} \frac{a^2\ \sin^2\theta}{\rho^2} & -\left(\frac{\rho^2+a^2\sin^2\theta}{\rho^2}\right) & \frac{a}{\rho^2} \\ -\left(\frac{\rho^2+a^2\sin^2\theta}{\rho^2}\right) & \frac{r^2+a^2-2mr}{\rho^2} & -\frac{a}{\rho^2} \\ \frac{a}{\rho^2} & -\frac{a}{\rho^2} & \frac{1}{\rho^2\sin^2\theta} \end{vmatrix}$$

$$
\begin{aligned}
&= \frac{a^2\ \sin^2\theta}{\rho^2}\begin{vmatrix} \frac{r^2+a^2-2mr}{\rho^2} & -\frac{a}{\rho^2} \\ -\frac{a}{\rho^2} & \frac{1}{\rho^2\sin^2\theta} \end{vmatrix} \\
&\quad + \left(\frac{\rho^2+a^2\sin^2\theta}{\rho^2}\right)\begin{vmatrix} -\left(\frac{\rho^2+a^2\sin^2\theta}{\rho^2}\right) & -\frac{a}{\rho^2} \\ \frac{a}{\rho^2} & \frac{1}{\rho^2\sin^2\theta} \end{vmatrix} \\
&\quad + \frac{a}{\rho^2}\begin{vmatrix} -\left(\frac{\rho^2+a^2\sin^2\theta}{\rho^2}\right) & \frac{r^2+a^2-2mr}{\rho^2} \\ \frac{a}{\rho^2} & -\frac{a}{\rho^2} \end{vmatrix} \\
&= \frac{a^2\ \sin^2\theta}{\rho^2}\left(\frac{r^2+a^2-2mr}{\rho^4\sin^2\theta} - \frac{a^2}{\rho^4}\right) \\
&\quad + \left(\frac{\rho^2+a^2\sin^2\theta}{\rho^2}\right)\left(-\frac{\rho^2+a^2\sin^2\theta}{\rho^4\sin^2\theta} + \frac{a^2}{\rho^4}\right) \\
&\quad + \frac{a}{\rho^2}\left(+a\frac{\rho^2+a^2\sin^2\theta}{\rho^4} - a\frac{r^2+a^2-2mr}{\rho^4}\right) \\
&= -\frac{(\rho^2+a^2\sin^2\theta)^2}{\rho^6\sin^2\theta} + \frac{a^2(\rho^2+a^2\sin^2\theta)}{\rho^6} + \frac{a^2}{\rho^6} \\
&= -\frac{1}{\rho^2\sin^2\theta} \qquad \text{(A.217)}
\end{aligned}
$$

Untuk komponen-23=komponen-32:

$$
-\begin{vmatrix} \frac{a^2\sin^2\theta}{\rho^2} & -\frac{\rho^2+a^2\sin^2\theta}{\rho^2} & 0 \\ -\frac{\rho^2+a^2\sin^2\theta}{\rho^2} & \frac{r^2+a^2-2mr}{\rho^2} & 0 \\ \frac{a}{\rho^2} & -\frac{a}{\rho^2} & 0 \end{vmatrix} = 0 \qquad \text{(A.218)}
$$

Untuk komponen-33:

$$\begin{vmatrix} \frac{a^2\sin^2\theta}{\rho^2} & -\frac{r^2+a^2}{\rho^2} & 0 \\ -\frac{r^2+a^2}{\rho^2} & \frac{r^2+a^2-2mr}{\rho^2} & 0 \\ 0 & 0 & \frac{1}{\rho^2} \end{vmatrix} = \frac{a^2\sin^2\theta}{\rho^2}\frac{r^2+a^2-2mr}{\rho^4}$$
$$+\frac{r^2+a^2}{\rho^2}\left(-\frac{r^2+a^2}{\rho^4}\right)$$
$$= \frac{1}{\rho^6}\left(a^2\sin^2\theta\left(r^2+a^2-2mr\right)\right)$$
$$-\frac{1}{\rho^6}\left(r^2+a^2\right)^2$$
$$= \frac{1}{\rho^6}\left(a^2\sin^2\theta\Delta-\left(r^2+a^2\right)^2\right) \tag{A.219}$$

dengan

$$\Delta = r^2+a^2-2mr \tag{A.220}$$

Sehingga tensor metrik kontravarian adalah:

$$g_{\mu\nu} = \frac{1}{|g^{\mu\nu}|}\mathrm{Adj}\{g^{\mu\nu}\} \tag{A.221}$$
$$= -\rho^4\sin^2\theta$$
$$\times \begin{vmatrix} \frac{\rho^2-2mr}{\rho^6\sin^2\theta} & \frac{1}{\rho^4\sin^2\theta} & 0 & \frac{2mra}{\rho^6} \\ \frac{1}{\rho^4\sin^2\theta} & 0 & 0 & -\frac{a}{\rho^4} \\ 0 & 0 & -\frac{1}{\rho^2\sin^2\theta} & 0 \\ \frac{2mra}{\rho^6} & -\frac{a}{\rho^4} & 0 & \frac{1}{\rho^6}\left(a^2\sin^2\theta\Delta-\left(r^2+a^2\right)^2\right) \end{vmatrix}$$

$$= \begin{pmatrix} -\left(1-\frac{2mr-Q^2}{\rho^2}\right) & -1 & 0 & -\frac{2mra\sin^2\theta}{\rho^2} \\ -1 & 0 & 0 & a\sin^2\theta \\ & 0 & \rho^2 & 0 \\ -\frac{2mra\sin^2\theta}{\rho^2} & a\sin^2\theta & 0 & -\frac{\sin^2\theta}{\rho^2}\left(a^2\sin^2\theta\Delta-\left(r^2+a^2\right)^2\right) \end{pmatrix}$$

Maka elemen garisnya adalah

$$\begin{aligned} ds^2 &= -\left(1-\frac{2mr-Q^2}{\rho^2}\right)du^2 - 2du\,dr - \sin^2\frac{4mra\theta}{\rho^2}du\,d\phi \\ &\quad +2a\sin^2\theta dr\,d\phi + \rho^2 d\theta^2 - \frac{\sin^2\theta}{\rho^2}\left(a^2\sin^2\theta\Delta-\left(r^2+a^2\right)d\phi^2\right. \end{aligned} \tag{A.222}$$

## A.8 Solusi Kerr-Newman

Untuk mempersingkat penulisan, diambil konstanta c=1.
Dari metrik Reissner-Nordstrom:

$$ds^2 = -\left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right)dt^2 + \left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right)^{-1}dr^2 + r^2 d\Omega^2 \tag{A.223}$$

untuk menghilangkan singularitas di $r = 2m$, maka diperkenalkan koordinat waktu yang baru yaitu $cu = ct - r^*$ sehingga

$$\begin{aligned} t &= u + r^* \\ dt^2 &= du^2 + 2du\,dr^* + dr^{*2} \end{aligned} \tag{A.224}$$

$r^*$ adalah tortoise koordinat yang memenuhi hubungan

$$dr = \left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right)dr^* \tag{A.225}$$

kemudian pers. dan pers. dimasukkan ke dalam pers.1 sehingga

$$
\begin{aligned}
ds^2 &= -\left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right)dt^2 \\
&\quad +\left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right)^{-1}\left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right)^2 dr^{*2}+r^2 d\Omega^2 \\
&= -\left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right)dt^2 \\
&\quad +\left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right)dr^{*2}+r^2 d\Omega^2 \\
&= -\left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right)\left(dt^2-dr^{*2}\right)+r^2 d\Omega^2 \\
&= -\left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right)\left(du^2+2du\,dr^*+dr^{*2}-dr^{*2}\right)+r^2 d\Omega^2 \\
&= -\left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right)\left(du^2+2du\,dr^*\right)+r^2 d\Omega^2 \\
&= -\left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right)2du^2 \\
&\quad -\left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right)2du\left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right)^{-1}dr+r^2 d\Omega^2 \\
&= -\left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right)du^2-2du\,dr+r^2 d\Omega^2 \qquad \text{(A.226)}
\end{aligned}
$$

sehingga tensor metrik dari metrik di atas adalah:

$$
g_{\mu\nu}=\begin{pmatrix}
-\left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right) & -1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & r^2 & 0 \\
0 & 0 & 0 & r^2\sin^2\theta
\end{pmatrix} \qquad \text{(A.227)}
$$

bentuk kontravarian dari tensor metrik di atas dapat dicari dengan cara:

$$\begin{aligned}
g_{\mu\nu}g^{\mu\nu} &= 1 \\
g_{\mu\nu}^{-1}g_{\mu\nu}g^{\mu\nu} &= g_{\mu\nu}^{-1} \\
g^{\mu\nu} &= g_{\mu\nu}^{-1} \\
g^{\mu\nu} &= \frac{1}{|g_{\mu\nu}|}\mathrm{Adj}\{g_{\mu\nu}\}
\end{aligned} \tag{A.228}$$

dengan $|g_{\mu\nu}|$ adalah:

$$\begin{aligned}
|g_{\mu\nu}| &= \begin{vmatrix} -\left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right) & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{vmatrix} \\
&= -\left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right)\begin{vmatrix} 0 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{vmatrix} \\
&\quad -(-1)\begin{vmatrix} -1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{vmatrix} \\
&= 0+1\begin{vmatrix} -1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{vmatrix} \\
&= -1\begin{vmatrix} r^2 & 0 \\ 0 & r^2\sin^2\theta \end{vmatrix} \\
&= -r^4\sin^2\theta
\end{aligned} \tag{A.229}$$

Serta $\mathrm{Adj}\{g_{\mu\nu}\}$ adalah sebagai berikut:
Untuk komponen-00, maka:

$$\begin{vmatrix} 0 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{vmatrix} = 0$$

Untuk komponen-01 = 10:

$$-\begin{vmatrix} -1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{vmatrix} = -1(-1)\begin{vmatrix} r^2 & 0 \\ 0 & r^2\sin^2\theta \end{vmatrix}$$
$$= r^4\sin^2\theta$$

Untuk komponen-02 = 20:

$$\begin{vmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{vmatrix} = 0$$

Untuk komponen-03 = 30:

$$-\begin{vmatrix} -1 & 0 & 0 \\ 0 & 0 & r^2 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

Untuk komponen-11:

$$\begin{vmatrix} -\left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right) & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{vmatrix} = -\left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right)r^4\sin^2\theta$$

Untuk komponen-12 = 21:

$$-\begin{vmatrix} -\left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right) & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{vmatrix} = 0$$

Untuk komponen-13 = 31:

$$\begin{vmatrix} -\left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right) & -1 & 0 \\ 0 & 0 & r^2 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

Untuk komponen-22:

$$\begin{aligned}\begin{vmatrix} -\left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right) & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{vmatrix} &= -\left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right)\begin{vmatrix} 0 & 0 \\ 0 & r^2\sin^2\theta \end{vmatrix} \\ &\quad -(-1)\begin{vmatrix} -c & 0 \\ 0 & r^2\sin^2\theta \end{vmatrix} \\ &= -r^2\sin^2\theta \end{aligned} \quad \text{(A.230)}$$

Untuk komponen-23 $=$ 32:

$$-\begin{vmatrix} -\left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right) & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

Untuk komponen-33:

$$\begin{aligned}&\begin{vmatrix} -\left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right) & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & r^2 \end{vmatrix} \\ &= -\left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right)\begin{vmatrix} 0 & 0 \\ 0 & r^2 \end{vmatrix} \\ &\quad -(-1)\begin{vmatrix} -1 & 0 \\ 0 & r^\theta \end{vmatrix} \\ &= -r^2 \end{aligned}$$

Sehingga tensor metrik kontravarian $g^{\mu\nu}$ menjadi:

$$\begin{aligned} g^{\mu\nu} &= \frac{1}{|g_{\mu\nu}|}\text{Adj}\{g^{\mu\nu}\} \\ &= \frac{1}{-r^4\sin^2\theta} \\ &\begin{pmatrix} 0 & r^4\sin^2\theta & 0 & 0 \\ r^4\sin^2\theta & \left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right)r^4\sin^2\theta & 0 & 0 \\ 0 & 0 & -r^2\sin^2\theta & 0 \\ 0 & 0 & 0 & -r^2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & \left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right) & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix} \end{aligned} \quad (A.231)$$

karena elemen garis dalam pernyataan tensor metrik adalah

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu \\ &= -l_\mu n_\nu - l_\nu n_\mu + m_\mu\bar{m}_\nu + m_\nu\bar{m}_\mu \end{aligned}$$

kemudian dilakukan $\nu \leftrightarrow \mu$ pada suku ke-2 dan ke-4 sehingga:

$$ds^2 = -2l_\mu n_\nu + 2m_\mu\bar{m}_\nu \quad (A.232)$$

Kemudian dipilih suku-suku vektor-4 null dari metrik Reissner Nordstrom diatas. Pemilihan ini bebas asalkan memenuhi sifat-sifat vektor-4 null yang telah didefinisikan. Dari metrik Reissner-Nordstrom:

$$\begin{aligned} ds^2 &= -\left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right)du^2 - 2du\,dr + r^2d\Omega^2 \\ &= -\left[\left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right)du + 2dr\right]du \\ &\quad +r\left(d\theta + i\ \sin\theta\ d\phi\right)r\left(d\theta - i\ \sin\theta\ d\phi\right) \end{aligned} \quad (A.233)$$

sehingga pemilihan komponen vektor-4 null adalah:

$$l_\mu dx^\mu = du \tag{A.234}$$

sehingga $l_\mu = (1, 0, 0, 0)$.
Kemudian untuk komponen $n_\mu$:

$$\begin{aligned} -2n_\mu dx^\mu &= -\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right) du + 2dr \\ n_\mu dx^\mu &= \frac{1}{2}\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right) du + dr \end{aligned} \tag{A.235}$$

sehingga $n_\mu = \left(\frac{1}{2}\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right), 1, 0, 0\right)$.
Untuk komponen $m_\mu$:

$$\begin{aligned} \sqrt{2} m_\mu dx^\mu &= r\,(d\theta + i\ \sin\theta\ d\phi) \\ &= \frac{r}{\sqrt{2}}\,(d\theta + i\ \sin\theta\ d\phi) \end{aligned} \tag{A.236}$$

Sehingga $m_\mu = (0, 0, 1, i\ \sin\theta)$.
$\bar{m}_\mu$ adalah kompleks konjugat dari $m_\mu$ sehingga $\bar{m}_\mu = (0, 0, 1, -i\ \sin\theta)$.

Maka kompenen-komponen vektor-4 null yang telah dipilih adalah:

$$\begin{aligned} l_\mu &= (1, 0, 0, 0) \\ n_\mu &= \left(\frac{1}{2}\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right), 1, 0, 0\right) \\ m_\mu &= (0, 0, 1, i\ \sin\theta) \\ \bar{m}_\mu &= (0, 0, 1, -i\ \sin\theta) \end{aligned} \tag{A.237}$$

bentuk kontravarian dari masing-masing komponen vektor-4 null di atas dapat dicari menggunakan tensor metrik kontravarian:

$$Z^\mu = g^{\mu\nu} Z_\nu \tag{A.238}$$

sehingga

$$l^\mu = g^{\mu\nu} l_\nu \tag{A.239}$$

nilai untuk masing-masing $\mu = (0, 1, 2, 3)$ adalah:
untuk $\mu = 0$

$$\begin{aligned} l^0 &= g^{0\nu} l_\nu \\ &= g^{00} l_0 + g^{01} l_1 + g^{02} l_2 + g^{03} l_3 \\ &= 0 \end{aligned}$$

untuk $\mu = 1$

$$\begin{aligned} l^1 &= g^{1\nu} l_\nu \\ &= g^{10} l_0 + g^{11} l_1 + g^{12} l_2 + g^{13} l_3 \\ &= -1 + 0 + 0 + 0 \\ &= 1 \end{aligned}$$

untuk $\mu = 2$

$$\begin{aligned} l^2 &= g^{2\nu} l_\nu \\ &= g^{20} l_0 + g^{21} l_1 + g^{22} l_2 + g^{23} l_3 \\ &= 0 \end{aligned}$$

untuk $\mu = 3$

$$\begin{aligned} l^3 &= g^{3\nu} l_\nu \\ &= g^{30} l_0 + g^{31} l_1 + g^{32} l_2 + g^{33} l_3 \\ &= 0 \end{aligned}$$

sehingga didapatkan:

$$l^\mu = (0, -1, 0, 0) \tag{A.240}$$

Kemudian untuk $n^\mu$ dengan masing-masing $\mu = (0, 1, 2, 3)$ adalah:

$$n^\mu = g^{\mu\nu} n_\nu \tag{A.241}$$

untuk $\mu = 0$

$$\begin{aligned} n^0 &= g^{0\nu} n_\nu \\ &= g^{00} n_0 + g^{01} n_1 + g^{02} n_2 + g^{03} n_3 \\ &= 0 + (-1).1 + 0 + 0 \\ &= -1 \end{aligned} \tag{A.242}$$

untuk $\mu = 1$

$$\begin{aligned} n^1 &= g^{1\nu} n_\nu \\ &= g^{10} n_0 + g^{11} n_1 + g^{12} n_2 + g^{13} n_3 \\ &= (-1)\frac{1}{2}\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right) + \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right).1 + 0 + 0 \\ &= \frac{1}{2}\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right) \end{aligned} \tag{A.243}$$

untuk $\mu = 2$

$$\begin{aligned} n^2 &= g^{2\nu} n_\nu \\ &= g^{20} n_0 + g^{21} n_1 + g^{22} n_2 + g^{23} n_3 \\ &= 0 \end{aligned} \tag{A.244}$$

untuk $\mu = 3$

$$\begin{aligned} n^3 &= g^{3\nu} n_\nu \\ &= g^{30} n_0 + g^{31} n_1 + g^{32} n_2 + g^{33} n_3 \\ &= 0 \end{aligned} \tag{A.245}$$

untuk $\mu = 4$

$$\begin{aligned} n^4 &= g^{4\nu} n_\nu \\ &= g^{40} n_0 + g^{41} n_1 + g^{42} n_2 + g^{43} n_3 \\ &= 0 \end{aligned} \tag{A.246}$$

sehingga didapatkan:

$$n^\mu = \left(-1, \frac{1}{2}\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right), 0, 0\right) \tag{A.247}$$

Kemudian untuk $m^\mu$ dengan masing-masing $\mu = (0, 1, 2, 3)$ adalah:

$$m^\mu = g^{\mu\nu} m_\nu \tag{A.248}$$

untuk $\mu = 0$

$$\begin{aligned} m^0 &= g^{0\nu} m_\nu \\ &= g^{00} m_0 + g^{01} m_1 + g^{02} m_2 + g^{03} m_3 \\ &= 0 \end{aligned}$$

untuk $\mu = 1$

$$\begin{aligned} m^1 &= g^{1\nu} m_\nu \\ &= g^{10} m_0 + g^{11} m_1 + g^{12} m_2 + g^{13} m_3 \\ &= 0 \end{aligned}$$

untuk $\mu = 2$

$$\begin{aligned} m^2 &= g^{2\nu} m_\nu \\ &= g^{20} m_0 + g^{21} m_1 + g^{22} m_2 + g^{23} m_3 \\ &= 0 + 0 + \frac{1}{r^2}\frac{r}{\sqrt{2}} + 0 \\ &= \frac{1}{r\sqrt{2}} \end{aligned}$$

untuk $\mu = 3$

$$\begin{aligned} m^3 &= g^{3\nu} m_\nu \\ &= g^{30} m_0 + g^{31} m_1 + g^{32} m_2 + g^{33} m_3 \\ &= 0 + 0 + 0 + \frac{1}{r^2 \sin^2 \theta} \frac{ir \ \sin \theta}{\sqrt{2}} \\ &= \frac{i}{r\sqrt{2} \sin \theta} \end{aligned}$$

sehingga didapatkan:

$$m^\mu = \frac{1}{r\sqrt{2}}(0, 0, 1, \frac{i}{\sin \theta}) \tag{A.249}$$

$\bar{m}$ adalah kompleks ku=onjugat dari $m$ sehingga:

$$\bar{m}^\mu = \frac{1}{r\sqrt{2}}(0, 0, 1, -\frac{i}{\sin \theta}) \tag{A.250}$$

Maka bentuk kontravarian dari komponen vektor-4 null adalah

$$\begin{aligned} l^\mu &= (0, -1, 0, 0) \\ n^\mu &= (-1, \frac{1}{2} \left( 1 - \frac{2m}{r} + \frac{Q^2}{r^2} \right), 0, 0) \\ m^\mu &= \frac{1}{r\sqrt{2}}(0, 0, 1, \frac{i}{\sin \theta}) \\ \bar{m}^\mu &= \frac{1}{r\sqrt{2}}(0, 0, 1, \frac{i}{\sin \theta}) \end{aligned} \tag{A.251}$$

atau

$$\begin{aligned}
l^{\mu}\partial_{\mu} &= -\partial_r \\
n^{\mu}\partial_{\mu} &= -\partial_u + \frac{1}{2}\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)\partial_r \\
m^{\mu}\partial_{\mu} &= \frac{1}{r\sqrt{2}}\left(\partial_\theta + \frac{i}{\sin\theta}\partial_\phi\right) \\
m^{\mu}\partial_{\mu} &= \frac{1}{r\sqrt{2}}\left(\partial_\theta - \frac{i}{\sin\theta}\partial_\phi\right)
\end{aligned} \tag{A.252}$$

Kemudian nilai $r$ dijadikan dalam bentuk kompleks sehingga:

$$\begin{aligned}
l^{\mu} &= (0, -1, 0, 0) \\
n^{\mu} &= (-1, \frac{1}{2}\left(1 - \frac{m}{r} - \frac{m}{\bar{r}} + \frac{Q^2}{r\bar{r}}\right), 0, 0) \\
m^{\mu} &= \frac{1}{\bar{r}\sqrt{2}}(0, 0, 1, \frac{i}{\sin\theta}) \\
\bar{m}^{\mu} &= \frac{1}{r\sqrt{2}}(0, 0, 1, \frac{i}{\sin\theta})
\end{aligned} \tag{A.253}$$

atau

$$\begin{aligned}
l^{\mu}\partial_{\mu} &= -\partial_r \\
n^{\mu}\partial_{\mu} &= -\partial_u + \frac{1}{2}\left(1 - \frac{m}{r} - \frac{m}{\bar{r}} + \frac{Q^2}{r\bar{r}}\right)\partial_r \\
m^{\mu}\partial_{\mu} &= \frac{1}{\bar{r}\sqrt{2}}\left(\partial_\theta + \frac{i}{\sin\theta}\partial_\phi\right) \\
\bar{m}^{\mu}\partial_{\mu} &= \frac{1}{r\sqrt{2}}\left(\partial_\theta - \frac{i}{\sin\theta}\partial_\phi\right)
\end{aligned} \tag{A.254}$$

setelah itu dilakukan transformasi

$$x^\rho \to x'^\rho = x^\rho + ia\cos\theta(\delta_0^\rho - \delta_1^\rho) \tag{A.255}$$

dengan a adalah suatu konstanta yang akan ditentukan kemudian sehingga:

$$\begin{aligned} u \to u' &= u + ia\ \cos\theta \\ r \to r' &= r - ia\ \cos\theta \\ \theta \to \theta' &= \theta \\ \phi \to \phi' &= \phi \end{aligned} \tag{A.256}$$

atau

$$\begin{aligned} u &= u' - ia\ \cos\theta = u' - ia\ \cos\theta' \\ r &= r' + ia\ \cos\theta = r' + ia\ \cos\theta' \end{aligned} \tag{A.257}$$

dengan $v', r', \theta', \phi'$ adalah kuantitas riil. Vektor-vektor basis bertransformasi dengan menggunakan aturan rantai:

$$\frac{\partial}{\partial x^\mu} \to \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu}$$

atau dapat dituliskan :

$$\partial_\mu \to \partial_{\mu'} = \frac{\partial x^\nu}{\partial x'^\mu}\partial_\nu \tag{A.258}$$

sehingga untuk masing-masing $\mu = (0, 1, 2, 3)$ didapatkan:

$$\begin{aligned} \partial_0 \to \partial_{0'} &= \frac{\partial x^\nu}{\partial x'^0}\partial_\nu \\ \partial_{u'} &= \frac{\partial x^\nu}{\partial u'}\partial_\nu \\ &= \frac{\partial u}{\partial u'}\partial_u + \frac{\partial r}{\partial u'}\partial_r + \frac{\partial \theta}{\partial u'}\partial_\theta + \frac{\partial \phi}{\partial u'}\partial_\phi \\ &= \partial_u \end{aligned} \tag{A.259}$$

$$\begin{aligned}
\partial_1 \to \partial_{1'} &= \frac{\partial x^\nu}{\partial x'^1}\partial_\nu \\
\partial_{r'} &= \frac{\partial x^\nu}{\partial r'}\partial_\nu \\
&= \frac{\partial u}{\partial r'}\partial_u + \frac{\partial r}{\partial r'}\partial_r + \frac{\partial \theta}{\partial r'}\partial_\theta + \frac{\partial \phi}{\partial r'}\partial_\phi \\
&= \partial_r
\end{aligned} \tag{A.260}$$

$$\begin{aligned}
\partial_2 \to \partial_{2'} &= \frac{\partial x^\nu}{\partial x'^2}\partial_\nu \\
\partial_{\theta'} &= \frac{\partial x^\nu}{\partial \theta'}\partial_\nu \\
&= \frac{\partial u}{\partial \theta'}\partial_u + \frac{\partial r}{\partial \theta'}\partial_r + \frac{\partial \theta}{\partial \theta'}\partial_\theta + \frac{\partial \phi}{\partial \theta'}\partial_\phi \\
&= ia\ \sin\theta'\ \partial_u - ia\ \sin\theta'\ \partial_r + \partial_\theta
\end{aligned} \tag{A.261}$$

$$\begin{aligned}
\partial_3 \to \partial_{3'} &= \frac{\partial x^\nu}{\partial x^3}\partial_\nu \\
\partial_{\phi'} &= \frac{\partial x^\nu}{\partial \phi'}\partial_\nu \\
&= \frac{\partial u}{\partial \phi'}\partial_u + \frac{\partial r}{\partial \phi'}\partial_r + \frac{\partial \theta}{\partial \phi'}\partial_\theta + \frac{\partial \phi}{\partial \phi'}\partial_\phi \\
&= \partial_\phi
\end{aligned} \tag{A.262}$$

Maka komponen-komponen vektor-4 null menjadi:

$$l^\mu \partial_\mu = -\partial_r \to l'^\mu \partial_{\mu'} = -\partial_{r'} = -\partial_r \tag{A.263}$$

$$
\begin{aligned}
n^\mu \partial_\mu \to n'^\mu \partial_{\mu'} &= -\partial_{u'} + \frac{1}{2}\left(1 - \frac{m}{r'} - \frac{m}{\bar{r'}} + \frac{Q^2}{r'\bar{r'}}\right)\partial_{r'} \\
&= -\partial_u + \frac{1}{2}\left(1 - \frac{m}{r' + ia\ \cos\theta'} - \frac{m}{r' - ia\ \cos\theta'}\right)\partial_r \\
&\quad + \frac{1}{2}\left(\frac{Q^2}{(r' + ia\ \cos\theta')(r' - ia\ \cos\theta')}\right)\partial_{r'} \\
&= -\partial_u + \frac{1}{2}\left(1 - \frac{2mr' - Q^2}{r'^2 + a^2\cos^2\theta'}\right)\partial_r \qquad \text{(A.264)}
\end{aligned}
$$

$$
\begin{aligned}
m^\mu \partial_\mu \to m'^\mu \partial_{\mu'} &= \frac{1}{\bar{r}\sqrt{2}}\left(\partial_{\theta'} + \frac{i}{\sin\theta}\partial_{\phi'}\right) \\
&= \frac{1}{\sqrt{2}(r' - ia\ \cos\theta')}\left(ia\ \sin\theta'\partial_{v'} - ia\ \sin\theta'\partial_{r'}\right. \\
&\quad \left. + \partial_{\theta'} + \frac{i}{\sin\theta'}\partial_{\phi'}\right)
\end{aligned}
\qquad \text{(A.265)}
$$

$$
\begin{aligned}
\bar{m}^\mu \partial_\mu \to \bar{m'}^\mu \partial_{\mu'} &= \frac{1}{r\sqrt{2}}\left(\partial_\theta - \frac{i}{\sin\theta}\partial_\phi\right) \\
&= \frac{1}{\sqrt{2}(r' + ia\ \cos\theta')}\left(-ia\ \sin\theta'\partial_{v'} + ia\ \sin\theta'\partial_{r'}\right. \\
&\quad \left. + \partial_{\theta'} - \frac{i}{\sin\theta'}\partial_{\phi'}\right)
\end{aligned}
\qquad \text{(A.266)}
$$

dengan menghilangkan tanda $'$ pada persamaan-persamaan di atas, maka diperoleh:

$$\begin{aligned} l^\mu\partial_\mu &= -\partial_r \\ n^\mu\partial_\mu &= -\partial_u + \frac{1}{2}\left(1 - \frac{2mr - Q^2}{r^2 + a^2\cos^2\theta}\right)\partial_r \\ m^\mu\partial_\mu &= \frac{1}{\sqrt{2}(r - ia\ \cos\theta)}\left(ia\ \sin\theta\partial_v - ia\ \sin\theta\partial_r + \partial_\theta + \frac{i}{\sin\theta}\partial_\phi\right) \\ \bar{m}^\mu\partial_\mu &= \frac{1}{\sqrt{2}(r + ia\ \cos\theta)}\left(-ia\ \sin\theta\partial_v + ia\ \sin\theta\partial_r + \partial_\theta - \frac{i}{\sin\theta}\partial_\phi\right) \end{aligned} \tag{A.267}$$

atau

$$\begin{aligned} l^\mu &= (0, -1, 0, 0) \\ n^\mu &= (-1, \frac{1}{2}\left(1 - \frac{2mr - Q^2}{r^2 + a^2\cos^2\theta}\right), 0, 0) \\ m^\mu &= \frac{1}{\sqrt{2}(r - ia\ \cos\theta)}\left(ia\ \sin\theta, -ia\ \sin\theta, 1, \frac{i}{\sin\theta}\right) \\ \bar{m}^\mu &= \frac{1}{\sqrt{2}(r + ia\ \cos\theta)}\left(-ia\ \sin\theta, ia\ \sin\theta, 1, -\frac{i}{\sin\theta}\right) \end{aligned} \tag{A.268}$$

Selanjutnya komponen-komponen tensor metrik kontravarian $g^{\mu\nu}$ dapat dibentuk yaitu:

$$g^{\mu\nu} = -l^\mu n^\nu - l^\mu n^\nu + m^\mu\bar{m}^\nu + m^\nu\bar{m}^\mu \tag{A.269}$$

masing-masing komponen tersebut adalah

$$
\begin{aligned}
g^{00} &= -l^0n^0 - l^0n^0 + m^0\bar{m}^0 + m^0\bar{m}^0 \\
&= -2l^0n^0 + 2m^0\bar{m}^0 \\
&= 0 + 2\frac{ia\ \sin\theta}{\sqrt{2}(r - ia\ \cos\theta)}.\frac{-ia\ \sin\theta}{\sqrt{2}(r + ia\ \cos\theta)} \\
&= \frac{a^2\ \sin^2\theta}{(r^2 + a^2\ \cos^2\theta)} \\
&= \frac{a^2\ \sin^2\theta}{\rho^2} \qquad \text{(A.270)}
\end{aligned}
$$

dengan

$$\rho^2 \equiv r^2 + a^2\cos^2\theta \qquad \text{(A.271)}$$

$$
\begin{aligned}
g^{01} &= g^{10} \\
&= -l^0n^1 - l^1n^0 + m^0\bar{m}^1 + m^1\bar{m}^0 \\
&= 0 - (-1).(-1) \\
&\quad + \frac{1}{\sqrt{2}(r - ia\ \cos\theta)}(ia\ \sin\theta).\frac{1}{\sqrt{2}(r + ia\ \cos\theta)}(ia\ \sin\theta) \\
&\quad + \frac{1}{\sqrt{2}(r + ia\ \cos\theta)}(-ia\ \sin\theta).\frac{1}{\sqrt{2}(r - ia\ \cos\theta)}(-ia\ \sin\theta) \\
&= -\left(1 + \frac{a^2\sin^2\theta}{r^2 + a^2\cos^2\theta}\right) \\
&= -\left(\frac{r^2 + a^2\cos^2\theta + a^2\sin^2\theta}{r^2 + a^2\cos^2\theta}\right) \\
&= -\left(\frac{r^2 + a^2}{r^2 + a^2\cos^2\theta}\right) \\
&= -\left(\frac{r^2 + a^2}{\rho^2}\right) \qquad \text{(A.272)}
\end{aligned}
$$

$$\begin{aligned}
g^{02} &= g^{20} \\
&= -l^0 n^2 - l^2 n^0 + m^0 \bar{m}^2 + m^2 \bar{m}^0 \\
&= 0 + \frac{ia\ \sin\theta}{\sqrt{2}(r - ia\ \cos\theta)} . \frac{1}{\sqrt{2}(r + ia\ \cos\theta)} \\
&\quad + \frac{1}{\sqrt{2}(r - ia\ \cos\theta)} . \frac{-ia\ \sin\theta}{\sqrt{2}(r + ia\ \cos\theta)} \\
&= 0
\end{aligned} \tag{A.273}$$

$$\begin{aligned}
g^{03} &= g^{30} \\
&= -l^0 n^3 - l^3 n^0 + m^0 \bar{m}^3 + m^3 \bar{m}^0 \\
&= 0 + \frac{ia\ \sin\theta}{\sqrt{2}(r - ia\ \cos\theta)} . \frac{-i}{\sqrt{2}\sin\theta(r + ia\ \cos\theta)} \\
&\quad + \frac{i}{\sqrt{2}\sin\theta(r - ia\ \cos\theta)} . \frac{-ia\ \sin\theta}{\sqrt{2}(r + ia\ \cos\theta)} \\
&= \frac{a}{r^2 + a^2\cos^2\theta} \\
&= \frac{a}{\rho^2}
\end{aligned} \tag{A.274}$$

$$\begin{aligned}
g^{11} &= -l^1 n^1 - l^1 n^1 + m^1 \bar{m}^1 + m^1 \bar{m}^1 \\
&= -2l^1 n^1 + 2m^1 \bar{m}^1 + \\
&= -2.(-1).\frac{1}{2}\left(1 - \frac{2mr - Q^2}{r^2 + a^2\cos^2\theta}\right) \\
&\quad +2.\frac{-ia\ \sin\theta}{\sqrt{2}(r - ia\ \cos\theta)}.\frac{ia\ \sin\theta}{\sqrt{2}(r + ia\ \cos\theta)} \\
&= 1 - \frac{2mr - Q^2}{r^2 + a^2\cos^2\theta} + \frac{a^2\sin^2\theta}{r^2 + a^2\cos^2\theta} \\
&= 1 - \frac{2mr - Q^2}{\rho^2} + \frac{a^2\sin^2\theta}{\rho^2} \\
&= \frac{\rho^2 - 2mr + Q^2 + a^2\sin^2\theta}{\rho^2} \\
&= \frac{r^2 + a^2\cos^2\theta - 2mr + Q^2 + a^2\sin^2\theta}{\rho^2} \\
&= \frac{r^2 + a^2 - 2mr + Q^2}{\rho^2}
\end{aligned} \tag{A.275}$$

$$\begin{aligned}
g^{12} &= g^{21} \\
&= -l^1 n^2 - l^2 n^1 + m^1 \bar{m}^2 + m^2 \bar{m}^1 \\
&= 0 + \frac{-ia\ \sin\theta}{\sqrt{2}(r - ia\ \cos\theta)}.\frac{1}{\sqrt{2}\sin\theta(r + ia\ \cos\theta)} \\
&\quad + \frac{1}{\sqrt{2}\sin\theta(r - ia\ \cos\theta)}.\frac{ia\ \sin\theta}{\sqrt{2}(r + ia\ \cos\theta)} \\
&= 0
\end{aligned} \tag{A.276}$$

$$
\begin{aligned}
g^{13} &= g^{31} \\
&= -l^1 n^3 - l^3 n^1 + m^1 \bar{m}^3 + m^3 \bar{m}^1 \\
&= 0 + \frac{-ia\ \sin\theta}{\sqrt{2}(r - ia\ \cos\theta)} . \frac{-i}{\sqrt{2}\sin\theta(r + ia\ \cos\theta)} \\
&\quad + \frac{i}{\sqrt{2}\sin\theta(r - ia\ \cos\theta)} . \frac{ia\ \sin\theta}{\sqrt{2}(r + ia\ \cos\theta)} \\
&= -\frac{a}{r^2 + a^2\cos^2\theta} \\
&= -\frac{a}{\rho^2} \qquad \text{(A.277)}
\end{aligned}
$$

$$
\begin{aligned}
g^{22} &= -l^2 n^2 - l^2 n^2 + m^2 \bar{m}^2 + m^2 \bar{m}^2 \\
&= -2l^2 n^2 + 2m^2 \bar{m}^2 \\
&= 0 + 2\frac{1}{\sqrt{2}(r - ia\ \cos\theta)} . \frac{1}{\sqrt{2}\sin\theta(r + ia\ \cos\theta)} \\
&= \frac{1}{r^2 + a^2\cos^2\theta} \\
&= \frac{1}{\rho^2} \qquad \text{(A.278)}
\end{aligned}
$$

$$
\begin{aligned}
g^{23} &= g^{32} \\
&= -l^2 n^3 - l^3 n^2 + m^2 \bar{m}^3 + m^3 \bar{m}^2 \\
&= 0 + \frac{1}{\sqrt{2}(r - ia\ \cos\theta)} . \frac{-i}{\sqrt{2}\sin\theta(r + ia\ \cos\theta)} \\
&\quad + \frac{i}{\sqrt{2}\sin\theta(r - ia\ \cos\theta)} . \frac{1}{\sqrt{2}(r + ia\ \cos\theta)} \\
&= 0 \qquad \text{(A.279)}
\end{aligned}
$$

$$
\begin{aligned}
g^{33} &= -l^3 n^3 - l^3 n^3 + m^3 \bar{m}^3 + m^3 \bar{m}^3 \\
&= -2l^3 n^3 + 2m^3 \bar{m}^3 \\
&= 0 + 2\frac{i}{\sqrt{2}\sin\theta(r - ia\ \cos\theta)} \cdot \frac{-i}{\sqrt{2}\sin\theta(r + ia\ \cos\theta)} \\
&= \frac{1}{\sin^2\theta(r^2 + a^2\cos^2\theta)} \\
&= \frac{1}{\rho^2\sin^2\theta} \qquad \text{(A.280)}
\end{aligned}
$$

Sehingga tensor metrik kontravarian adalah

$$
g^{\mu\nu} = \begin{pmatrix} \frac{a^2\ \sin^2\theta}{\rho^2} & -\left(\frac{r^2+a^2}{\rho^2}\right) & 0 & \frac{a}{\rho^2} \\ -\left(\frac{r^2+a^2}{\rho^2}\right) & \frac{r^2+a^2-2mr+Q^2}{\rho^2} & 0 & -\frac{a}{\rho^2} \\ 0 & 0 & \frac{1}{\rho^2} & 0 \\ \frac{a}{\rho^2} & -\frac{a}{\rho^2} & 0 & \frac{1}{\rho^2\sin^2\theta} \end{pmatrix} \qquad \text{(A.281)}
$$

karena $\rho^2 \equiv r^2 + a^2\cos^2\theta$ , maka:

$$
g^{\mu\nu} = \begin{pmatrix} \frac{a^2\ \sin^2\theta}{\rho^2} & -\left(\frac{\rho^2+a^2\sin^2\theta}{\rho^2}\right) & 0 & \frac{a}{\rho^2} \\ -\left(\frac{\rho^2+a^2\sin^2\theta}{\rho^2}\right) & \frac{r^2+a^2-2mr+Q^2}{\rho^2} & 0 & -\frac{a}{\rho^2} \\ 0 & 0 & \frac{1}{\rho^2} & 0 \\ \frac{a}{\rho^2} & -\frac{a}{\rho^2} & 0 & \frac{1}{\rho^2\sin^2\theta} \end{pmatrix} \qquad \text{(A.282)}
$$

Bentuk kovarian dari tensor metrik di atas adalah

$$
g_{\mu\nu} = \frac{1}{|g^{\mu\nu}|}\mathrm{Adj}\{g^{\mu\nu}\} \qquad \text{(A.283)}
$$

dengan

$$
\begin{aligned}
|g^{\mu\nu}| &= \begin{vmatrix} \frac{a^2\ \sin^2\theta}{\rho^2} & -\left(\frac{\rho^2+a^2\sin^2\theta}{\rho^2}\right) & 0 & \frac{a}{\rho^2} \\ -\left(\frac{\rho^2+a^2\sin^2\theta}{\rho^2}\right) & \frac{r^2+a^2-2mr+Q^2}{\rho^2} & 0 & -\frac{a}{\rho^2} \\ 0 & 0 & \frac{1}{\rho^2} & 0 \\ \frac{a}{\rho^2} & -\frac{a}{\rho^2} & 0 & \frac{1}{\rho^2\sin^2\theta} \end{vmatrix} \\
&= \frac{a^2\ \sin^2\theta}{\rho^2}\left[\frac{r^2+a^2-2mr+Q^2}{\rho^2}\left(\frac{1}{\rho^4\sin^2\theta}\right)\right] \\
&\quad + \frac{a^2\ \sin^2\theta}{\rho^2}\left(-\frac{a}{\rho^2}\right)\frac{a}{\rho^4} \\
&\quad + \frac{\rho^2+a^2\ \sin^2\theta}{\rho^2}\left[-\frac{\rho^2 a^2\ \sin^2\theta}{\rho^2}\left(\frac{1}{\rho^4\sin^2\theta}\right)\right] \\
&\quad + \frac{\rho^2+a^2\ \sin^2\theta}{\rho^2}\left[\left(-\frac{a}{\rho^2}\right)\frac{a}{\rho^4}\right] \\
&\quad + \left(-\frac{a}{\rho^2}\right)\left[-\frac{\rho^2+a^2\ \sin^2\theta}{\rho^2}\left(\frac{a}{\rho^4}\right)\right] \\
&\quad + \left(-\frac{a}{\rho^2}\right)\left[-\frac{r^2+a^2-2mr+Q^2}{\rho^2}\left(\frac{a}{\rho^4}\right)\right] \\
&= -\frac{1}{\rho^4\sin^2\theta} \qquad \text{(A.284)}
\end{aligned}
$$

Serta komponen-komponen Adj$\{g^{\mu\nu}\}$ adalah:

Untuk komponen-00:

$$\begin{vmatrix} \frac{r^2+a^2-2mr+Q^2}{\rho^2} & 0 & -\frac{a}{\rho^2} \\ 0 & \frac{1}{\rho^2} & 0 \\ -\frac{a}{\rho^2} & 0 & \frac{1}{\rho^2\sin^2\theta} \end{vmatrix} = \frac{r^2+a^2-2mr+Q^2}{\rho^2}\begin{vmatrix} \frac{1}{\rho^2} & 0 \\ 0 & \frac{1}{\rho^2\sin^2\theta} \end{vmatrix}$$

$$+\left(-\frac{a}{\rho^2}\right)\begin{vmatrix} 0 & \frac{1}{\rho^2} \\ -\frac{a}{\rho^2} & 0 \end{vmatrix}$$

$$= \frac{r^2+a^2-2mr+Q^2}{\rho^2}\frac{1}{\rho^2}\frac{1}{\rho^2\sin^2\theta}$$

$$+\left(-\frac{a}{\rho^2}\right)\left(\frac{1}{\rho^2}\frac{a}{\rho^2}\right)$$

$$= \frac{r^2+a^2-2mr+Q^2-a^2\sin^2\theta}{\rho^6\sin^2\theta}$$

$$= \frac{\rho^2-2mr+Q^2}{\rho^6\sin^2\theta} \tag{A.285}$$

Untuk komponen-01=komponen-10:

$$-\begin{vmatrix} -\frac{\rho^2+a^2\sin^2\theta}{\rho^2} & 0 & -\frac{a}{\rho^2} \\ 0 & \frac{1}{\rho^2} & 0 \\ \frac{a}{\rho^2} & 0 & \frac{1}{\rho^2\sin^2\theta} \end{vmatrix} = \frac{\rho^2+a^2\sin^2\theta}{\rho^2}\begin{vmatrix} \frac{1}{\rho^2} & 0 \\ 0 & \frac{1}{\rho^2\sin^2\theta} \end{vmatrix}$$

$$\frac{a}{\rho^2}\begin{vmatrix} 0 & \frac{1}{\rho^2} \\ \frac{a}{\rho^2} & 0 \end{vmatrix}$$

$$= \frac{\rho^2+a^2\sin^2\theta}{\rho^2}\frac{1}{\rho^4\sin^2\theta} - \frac{a^2}{\rho^6}$$

$$= \frac{1}{\rho^4\sin^2\theta} \tag{A.286}$$

Untuk komponen-02=komponen-20:

$$\begin{vmatrix} -\frac{\rho^2+a^2\sin^2\theta}{\rho^2} & \frac{r^2+a^2-2mr+Q^2}{\rho^2} & -\frac{a}{\rho^2} \\ 0 & 0 & 0 \\ \frac{a}{\rho^2} & \frac{a}{\rho^2} & \frac{1}{\rho^2\sin^2\theta} \end{vmatrix} = 0 \qquad \text{(A.287)}$$

Untuk komponen-03=komponen-30:

$$\begin{aligned} -\begin{vmatrix} -\frac{\rho^2+a^2\sin^2\theta}{\rho^2} & \frac{r^2+a^2-2mr+Q^2}{\rho^2} & 0 \\ 0 & 0 & \frac{1}{\rho^2} \\ \frac{a}{\rho^2} & -\frac{a}{\rho^2} & 0 \end{vmatrix} &= \frac{\rho^2+a^2\sin^2\theta}{\rho^2}\begin{vmatrix} 0 & \frac{1}{\rho^2} \\ -\frac{a}{\rho^2} & 0 \end{vmatrix} \\ &\quad +\frac{r^2+a^2-2mr+Q^2}{\rho^2}\left(-\frac{a}{\rho^4}\right) \\ &= a\frac{\rho^2+a^2\sin^2\theta}{\rho^6} \\ &\quad -a\frac{r^2+a^2-2mr+Q^2}{\rho^6} \\ &= a\frac{r^2+a^2}{\rho^6} \\ &\quad -a\frac{r^2+a^2-2mr+Q^2}{\rho^6} \\ &= \frac{(2mr-Q^2)a}{\rho^6} \qquad \text{(A.288)} \end{aligned}$$

Untuk komponen-11:

$$\begin{vmatrix} \frac{a^2\sin^2\theta}{\rho^2} & 0 & \frac{a}{\rho^2} \\ 0 & \frac{1}{\rho^2} & 0 \\ \frac{a}{\rho^2} & 0 & \frac{1}{\rho^2\sin^2\theta} \end{vmatrix} = \frac{a^2}{\rho^6} - \frac{a^2}{\rho^6} = 0 \qquad \text{(A.289)}$$

Untuk komponen-12=komponen-21:

$$-\begin{vmatrix} \frac{a^2\sin^2\theta}{\rho^2} & -\frac{\rho^2+a^2\sin^2\theta}{\rho^2} & \frac{a}{\rho^2} \\ 0 & 0 & 0 \\ \frac{a}{\rho^2} & -\frac{a}{\rho^2} & \frac{1}{\rho^2} \\ sin^2\theta & & \end{vmatrix} = 0 \tag{A.290}$$

Untuk komponen-13=komponen-31:

$$\begin{aligned} \begin{vmatrix} \frac{a^2\sin^2\theta}{\rho^2} & -\frac{\rho^2+a^2\sin^2\theta}{\rho^2} & 0 \\ 0 & 0 & \frac{1}{\rho^2} \\ \frac{a}{\rho^2} & -\frac{a}{\rho^2} & 0 \end{vmatrix} &= \frac{a^3\sin^2\theta}{\rho^6} \\ &\quad -a\frac{\rho^2+a^2\sin^2\theta}{\rho^6} \\ &= -\frac{a}{\rho^4} \end{aligned} \tag{A.291}$$

Untuk komponen-22;

$$\begin{vmatrix} \frac{a^2\ \sin^2\theta}{\rho^2} & -\left(\frac{\rho^2+a^2\sin^2\theta}{\rho^2}\right) & \frac{a}{\rho^2} \\ -\left(\frac{\rho^2+a^2\sin^2\theta}{\rho^2}\right) & \frac{r^2+a^2-2mr+Q^2}{\rho^2} & -\frac{a}{\rho^2} \\ \frac{a}{\rho^2} & -\frac{a}{\rho^2} & \frac{1}{\rho^2\sin^2\theta} \end{vmatrix}$$

$$
\begin{aligned}
&= \frac{a^2\ \sin^2\theta}{\rho^2}\begin{vmatrix} \frac{r^2+a^2-2mr+Q^2}{\rho^2} & -\frac{a}{\rho^2} \\ -\frac{a}{\rho^2} & \frac{1}{\rho^2\sin^2\theta} \end{vmatrix} \\
&\quad + \left(\frac{\rho^2+a^2\sin^2\theta}{\rho^2}\right)\begin{vmatrix} -\left(\frac{\rho^2+a^2\sin^2\theta}{\rho^2}\right) & -\frac{a}{\rho^2} \\ \frac{a}{\rho^2} & \frac{1}{\rho^2\sin^2\theta} \end{vmatrix} \\
&\quad + \frac{a}{\rho^2}\begin{vmatrix} -\left(\frac{\rho^2+a^2\sin^2\theta}{\rho^2}\right) & \frac{r^2+a^2-2mr+Q^2}{\rho^2} \\ \frac{a}{\rho^2} & -\frac{a}{\rho^2} \end{vmatrix} \\
&= \frac{a^2\ \sin^2\theta}{\rho^2}\left(\frac{r^2+a^2-2mr+Q^2}{\rho^4\sin^2\theta} - \frac{a^2}{\rho^4}\right) \\
&\quad + \left(\frac{\rho^2+a^2\sin^2\theta}{\rho^2}\right)\left(-\frac{\rho^2+a^2\sin^2\theta}{\rho^4\sin^2\theta} + \frac{a^2}{\rho^4}\right) \\
&\quad + \frac{a}{\rho^2}\left(+a\frac{\rho^2+a^2\sin^2\theta}{\rho^4} - a\frac{r^2+a^2-2mr+Q^2}{\rho^4}\right) \\
&= -\frac{(\rho^2+a^2\sin^2\theta)^2}{\rho^6\sin^2\theta} + \frac{a^2(\rho^2+a^2\sin^2\theta)}{\rho^6} + \frac{a^2}{\rho^6} \\
&= -\frac{1}{\rho^2\sin^2\theta}
\end{aligned}
\tag{A.292}
$$

Untuk komponen-23=komponen-32:

$$
-\begin{vmatrix} \frac{a^2\sin^2\theta}{\rho^2} & -\frac{\rho^2+a^2\sin^2\theta}{\rho^2} & 0 \\ -\frac{\rho^2+a^2\sin^2\theta}{\rho^2} & \frac{r^2+a^2-2mr+Q^2}{\rho^2} & 0 \\ \frac{a}{\rho^2} & -\frac{a}{\rho^2} & 0 \end{vmatrix} = 0 \tag{A.293}
$$

Untuk komponen-33:

$$\begin{vmatrix} \frac{a^2\sin^2\theta}{\rho^2} & -\frac{r^2+a^2}{\rho^2} & 0 \\ -\frac{r^2+a^2}{\rho^2} & \frac{r^2+a^2-2mr+Q^2}{\rho^2} & 0 \\ 0 & 0 & \frac{1}{\rho^2} \end{vmatrix} = \frac{a^2\sin^2\theta}{\rho^2}\frac{r^2+a^2-2mr+Q^2}{\rho^4}$$

$$+\frac{r^2+a^2}{\rho^2}\left(-\frac{r^2+a^2}{\rho^4}\right)$$

$$= \frac{1}{\rho^6}\left(a^2\sin^2\theta\left(r^2+a^2-2mr+Q^2\right)\right)$$

$$-\frac{1}{\rho^6}\left(r^2+a^2\right)^2$$

$$= \frac{1}{\rho^6}\left(a^2\sin^2\theta\Delta-\left(r^2+a^2\right)^2\right) \quad \text{(A.294)}$$

dengan

$$\Delta = r^2+a^2-2mr+Q^2 \quad \text{(A.295)}$$

Sehingga tensor metrik kontravarian adalah:

$$g_{\mu\nu} = \frac{1}{|g^{\mu\nu}|}\mathrm{Adj}\{g^{\mu\nu}\} \quad \text{(A.296)}$$

$$= -\rho^4\sin^2\theta$$

$$\times\begin{vmatrix} \frac{\rho^2-2mr+Q^2}{\rho^6\sin^2\theta} & \frac{1}{\rho^4\sin^2\theta} & 0 & \frac{(2mr-Q^2)a}{\rho^6} \\ \frac{1}{\rho^4\sin^2\theta} & 0 & 0 & -\frac{a}{\rho^4} \\ 0 & 0 & -\frac{1}{\rho^2\sin^2\theta} & 0 \\ \frac{(2mr-Q^2)a}{\rho^6} & -\frac{a}{\rho^4} & 0 & \frac{1}{\rho^6}\left(a^2\sin^2\theta\Delta-\left(r^2+a^2\right)^2\right) \end{vmatrix}$$

$$
= \begin{pmatrix}
-\left(1-\frac{2mr-Q^2}{\rho^2}\right) & -1 & 0 & -\frac{(2mr-Q^2)a\ \sin^2\theta}{\rho^2} \\
-1 & 0 & 0 & a\ \sin^2\theta \\
 & 0 & \rho^2 & 0 \\
-\frac{(2mr-Q^2)a\ \sin^2\theta}{\rho^2} & a\ \sin^2\theta & 0 & -\frac{\sin^2\theta}{\rho^2}\left(a^2\sin^2\theta\Delta-\left(r^2+a^2\right)^2\right)
\end{pmatrix}
$$

Maka elemen garisnya adalah

$$
\begin{aligned}
ds^2 \quad &= \quad -\left(1-\frac{2mr-Q^2}{\rho^2}\right)du^2-2du\,dr-2a\ \sin^2\frac{(2mr-Q^2)\theta}{\rho^2}du\,d\phi \\
&\quad +2a\ \sin^2\theta dr\ d\phi+\rho^2 d\theta^2-\frac{\sin^2\theta}{\rho^2}\left(a^2\sin^2\theta\Delta-\left(r^2+a^2\right)d\phi^2\right.
\end{aligned} \tag{A.297}
$$

Metrik di atas merupakan metrik Kerr-Newman.

*Halaman ini sengaja dikosongkan*

# BIODATA PENULIS

Andika Irawan, anak ke-2 dari 4 bersaudara, lahir di Lumajang, Jawa Timur pada 13 Juni 1993. Dibesarkan di lingkungan keluarga petani. Sejak kecil penulis mempunyai hobi renang, memancing, beladiri, mendaki gunung serta traveling. Penulis sangat membenci Fisika sampai pada saat-saat awal SMA yang kemudian bertemu guru yang telah ”meracuni” dan ”men-comblangkan” penulis sehingga Fisika menjadi pelajaran yang sangat disukai.

Riwayat pendidikan formal yang pernah ditempuh adalah SDN Pasrujambe 06 (1999-2005), SMPN 01 Pasrujambe (2005-2008), SMAN Candipuro (2008-2011) dan Jurusan Fisika ITS Surabaya (2012-2016). Tidak pernah merasakan pendidikan di Taman Kanak-Kanak dan selama menjadi siswa SD dan SMP penulis lebih banyak menghabiskan waktu untuk bermain dan menekuni hobi daripada belajar. Penulis sempat menerima beasiswa prestasi dari UPT Pendidikan Kec. Pasrujambe dan menjalani karantina di SDN Pasrujambe 01 selama 1 semester dengan 6 siswa lain dari SDN yang berbeda. Sewaktu SMA Penulis aktif di kegiatan Olimpiade Sains Nasional bidang Fisika serta aktif berorganisasi sebagai anggota OSIS, Bendahara Dewan Ambalan Gudep Jend. Sudirman Candipuro, Dewan Karja Ranting Candipuro, Pramuka SAKA Bhayangkara angkatan XIII Polres Lumajang serta PSHT ranting Candipuro.

Riwayat pelatihan manajemen yang pernah diikuti selama mahasiswa antara lain LKMM Pra-TD FMIPA ITS, LKMM TD Himasika ITS dan LOT PSDM BEM ITS. Organisasi yang pernah diikuti selama mahasiswa antara lain Himasika ITS (Departemen PSDM), BEM ITS (Kementrian PSDM), Fosif ITS (Divisi Kederisasi) dan UKM PSHT ITS (Divisi Kepelatihan).

Seminar ilmiah nasional dan internasional yang pernah diikuti antara lain *Kendaraan Massal Berbasis Listrik* (Kementrian Riset dan Teknologi), Jakarta, 2014, $4^{th}$ *Particle Physics School in South-East Asia*, VNU University of Science, Hanoi, Vietnam, September 2015, *Memorial Meeting for Nobel Laureate Prof Abdus Salam's 90th Birthday*, Institute of Advanced Studies, Nanyang Technological University, Singapore, Januari 2016.

Prestasi yang pernah dicapai selama mahasiswa antara lain *Honorable Mention ON-MIPA* (bidang Fisika) 2015, Juara 2 *Olimpiade Sains dan Teknologi Nasional* (bidang Fisika), Yogyakarta 2015, *Grand-Finalis OSN Pertamina* (bidang Fisika), Wisma Makara Universitas Indonesia, Jakarta 2015. Selama mahasiswa, penulis pernah menjadi asisten matakuliah Fisika Dasar 1&2, asisten Laboratorium Fisika Dasar 1&2 serta asisten matakuliah Fisika Matematika, serta sering memberikan tutorial untuk matakuliah Medan Elektromagnetik 1&2, Fisika Statistik dan Fisika Kuantum.